Impedance Review

ME 473

Professor Sawyer B. Fuller

Technically, "mechanical impedance" is $1/Z(s)$. Therefore, in the mechanical domain, we must refer to our definition of $Z(s)$ as "generalized impedance," or "mobility." This convention for $Z(s)$ has the benefit that it preserves circuit topology.

Generalized impedances are an extension of the concept of electrical impedances to systems of other domains. The table below lists the corresponding driving-point impedance definitions for five different energy modalities.

		Mechanical Translational	**Mechanical Rotational**	**Electrical**	**Fluid**	**Thermal**
Across Variable		v, velocity	ω, angular velocity	v, voltage	p, pressure	T, temperature
Through Variable		f, force	T, torque	i, current	q, volumetric flow	q, heat flow rate
Impedance $Z(s)$ **Admittance** $Y(s)=\dfrac{1}{Z(s)}$		$Z(s)=\dfrac{V(s)}{F(s)}$	$Z(s)=\dfrac{\Omega(s)}{T(s)}$	$Z(s)=\dfrac{V(s)}{I(s)}$	$Z(s)=\dfrac{P(s)}{Q(s)}$	$Z(s)=\dfrac{T(s)}{Q(s)}$
Impedance Z(s)	**A-Type**	mass, M: $\dfrac{1}{Ms}$	inertia, J: $\dfrac{1}{Js}$	capacitor, C $\dfrac{1}{Cs}$	fluid capacitor, C $\dfrac{1}{Cs}$	thermal capacitor, C $\dfrac{1}{Cs}$
	D-Type	damper, B $\dfrac{1}{B}$	r. damper, B $\dfrac{1}{B}$	resistor, R R	fluid resistor, R R	thermal resistor, R R
	T-Type	spring, K $\dfrac{s}{K}$	r. spring, K_r $\dfrac{s}{K_r}$	inductor, L Ls	fluid inductor, L Ls	—

$$\text{Impedance} \triangleq \frac{\text{Across Power Variable}}{\text{Through Power Variable}}$$

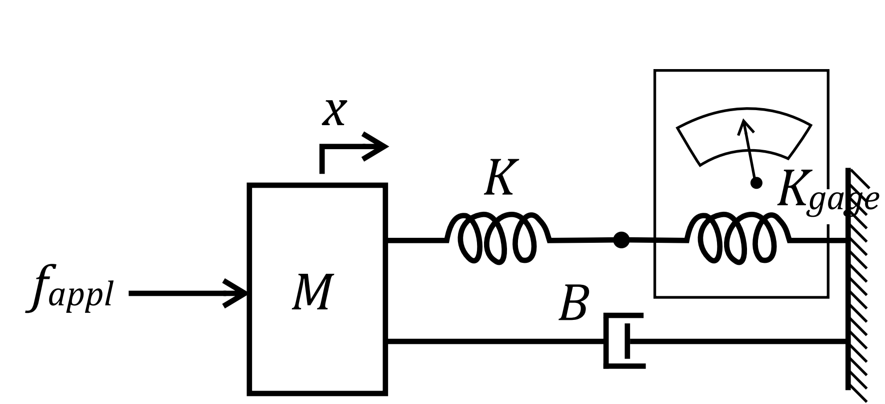
$$\text{Admittance} \triangleq \frac{\text{Through Power Variable}}{\text{Across Power Variable}}$$

Reference: Chapter 13 of Rowell & Wormley's *System Dynamics an Introduction*

$$\textit{Power} \triangleq \text{Through Var.} \times \text{Across Var.}$$

Standard Capacitor

Standard Resistor



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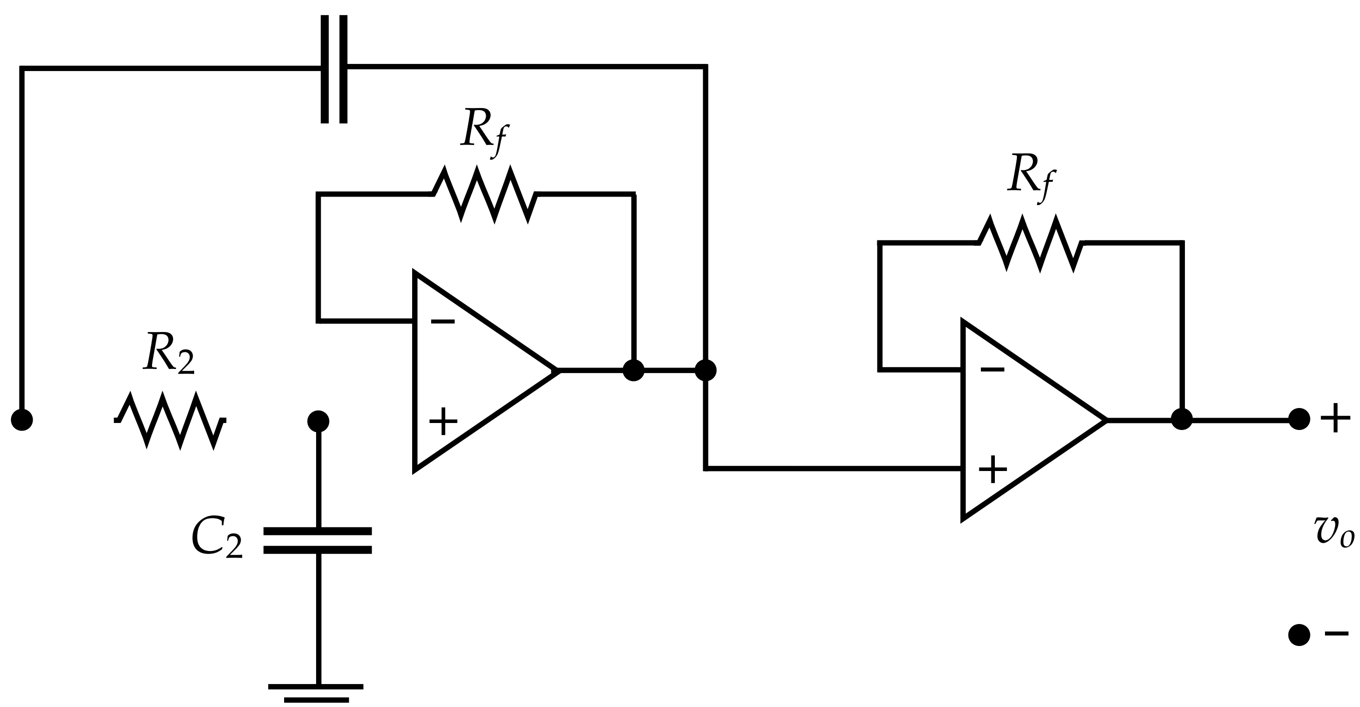

Generalized impedances are an extension of the concept of electrical impedances to systems of other domains.
The table below lists the corresponding driving-point impedance definitions for five different energy modalities.

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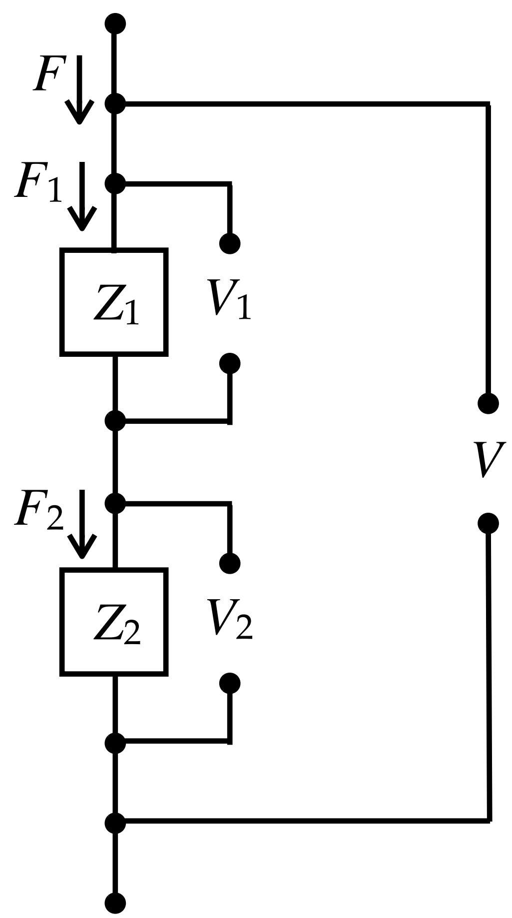
Impedance $Z(s)$ **Admittance** $Y(s)=\frac{1}{Z(s)}$		$Z(s)=\frac{V(s)}{F(s)}$	$Z(s)=\frac{\Omega(s)}{T(s)}$	$Z(s)=\frac{V(s)}{I(s)}$	$Z(s)=\frac{P(s)}{Q(s)}$	$Z(s)=\frac{T(s)}{Q(s)}$
Impedance Z(s)	**A-Type**	mass, M: $\frac{1}{Ms}$	inertia, J: $\frac{1}{Js}$	capacitor, C $\frac{1}{Cs}$	fluid capacitor, C $\frac{1}{Cs}$	thermal capacitor, C $\frac{1}{Cs}$
Impedance Z(s)	**D-Type**	damper, B $\frac{1}{B}$	r. damper, B $\frac{1}{B}$	resistor, R R	fluid resistor, R R	thermal resistor, R R
Impedance Z(s)	**T-Type**	spring, K $\frac{s}{K}$	r. spring, K_r $\frac{s}{K_r}$	inductor, L Ls	fluid inductor, L Ls	—

Following slides apply to all domains but use symbols from this domain

Reference: Chapter 13 of Rowell & Wormley's *System Dynamics an Introduction*

What about the impedance/admittance of a *connection* of passive elements?

Impedances Sum in Series

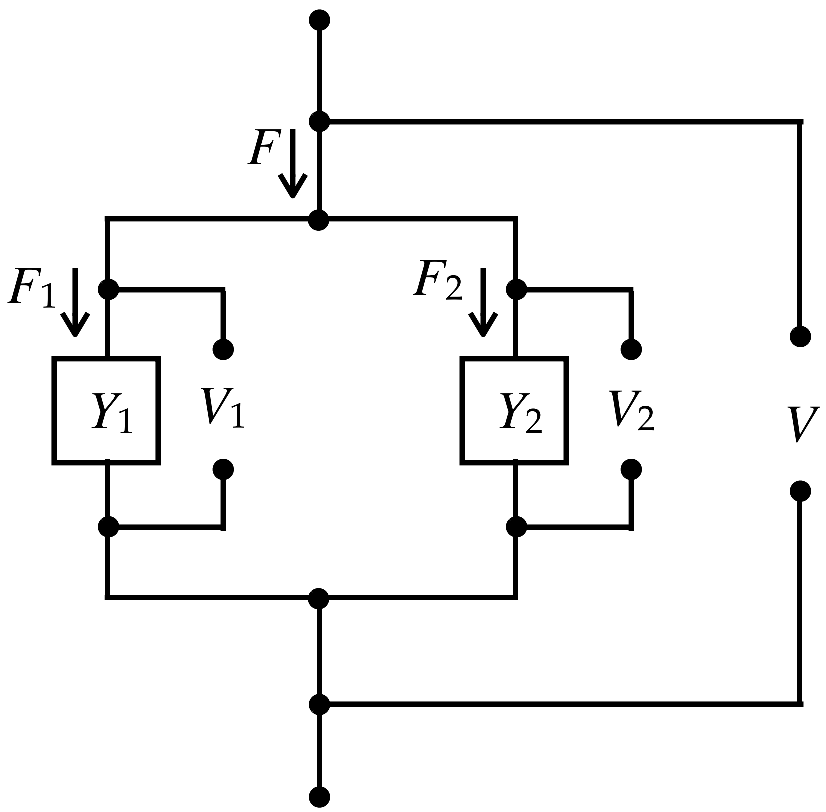


$$V = V_1 + V_2$$

$$F = F_1 = F_2$$

$$Z = \frac{V}{F} = \frac{V_1 + V_2}{F} = \frac{V_1}{F} + \frac{V_2}{F} = \frac{V_1}{F_1} + \frac{V_2}{F_2} = Z_1 + Z_2$$

admittances sum when they are in parallel



$$F = F_1 + F_2$$

$$V = V_1 = V_2$$

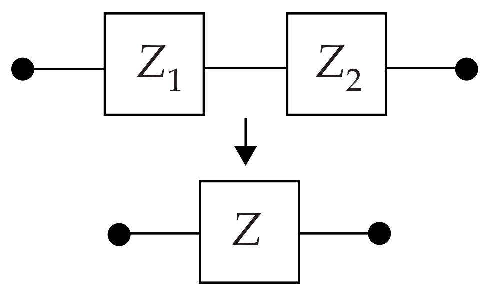
$$Y = \frac{F}{V} = \frac{F_1 + F_2}{V} = \frac{F_1}{V} + \frac{F_2}{V} = \frac{F_1}{V_1} + \frac{F_2}{V_2} = Y_1 + Y_2$$

Series and parallel combinations of impedances and admittances can be combined. In the following V and F represent the across and through variables respectively of any physical domain.

Series Combination

Elements sharing a common *through* variable are in *series*.

The *impedance* of elements connected in *series* is the sum of the individual *impedances*.



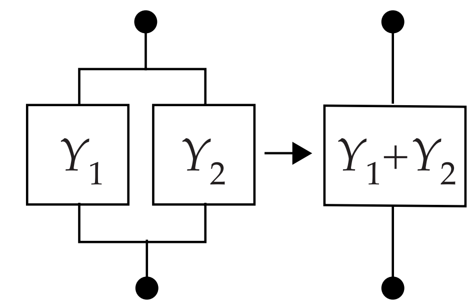
$$Z = Z_1 + Z_2$$

$$Y = \frac{1}{Z} = \frac{1}{Z_1 + Z_2} = \frac{Y_1 Y_2}{Y_1 + Y_2}$$

Parallel Combination

Elements sharing a common *across* variable are in *parallel*.

The *admittance* of elements connected in *parallel* is the sum of the individual *admittances*.



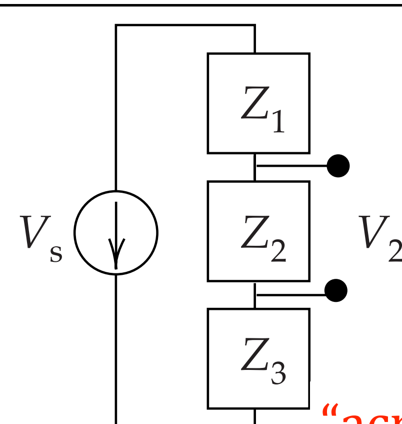
$$Y = Y_1 + Y_2$$

$$Z = \frac{1}{Y} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Simple transfer functions can be determined from impedance/admittance properties.

Across Variable Divider

The complex amplitude of the *across variable* across a set of elements in *series* is divided among the elements in proportion their *impedances*.

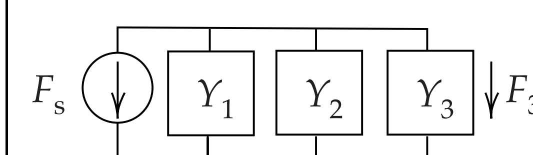


"across variable divider formula"

$$T(s) = \frac{V_2(s)}{V_s(s)} = \frac{Z_2}{Z_1 + Z_2 + Z_3}$$

Through Variable Divider

The complex amplitude of the *through variable* through a set of elements in *parallel* is divided among the elements in proportion their *admittances*.

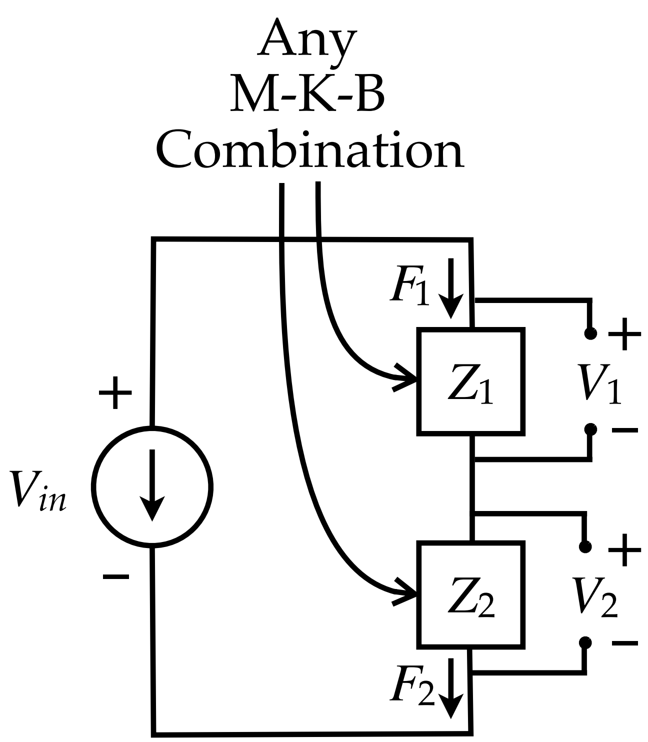


$$T(s) = \frac{F_2(s)}{F_s(s)} = \frac{Y_2}{Y_1 + Y_2 + Y_3}$$

"through variable divider formula"

Across Variable Divider

(Across Variable $= v$)



By inspection of the diagram

$$\frac{V_2(s)}{V_{in}(s)} = \frac{V_2(s)}{V_2(s)+V_1(s)} \tag{1}$$

and

$$F_1(s) = F_2(s) \tag{2}$$

Define the *impedances* $Z_1(s)$ and $Z_2(s)$ as

$$Z_1(s) = \frac{V_1(s)}{F_1(s)} \qquad Z_2(s) = \frac{V_2(s)}{F_2(s)} \tag{3}$$

From (1), using (3),

$$\frac{V_2(s)}{V_{in}(s)} = \frac{Z_2(s)F_2(s)}{Z_2(s)F_2(s)+Z_1(s)F_1(s)} \tag{4}$$

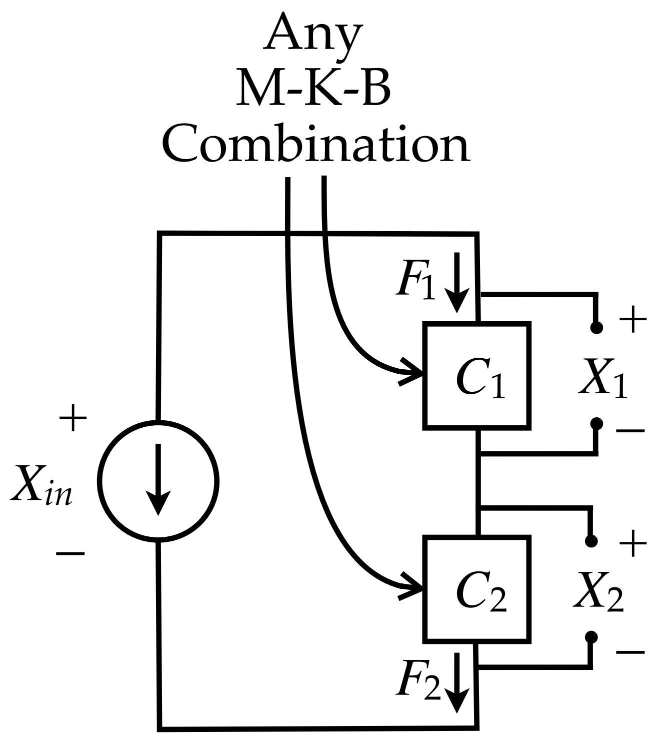
From (4), using (2),

$$\boxed{\frac{V_2(s)}{V_{in}(s)} = \frac{Z_2(s)}{Z_2(s)+Z_1(s)}}$$

"across variable divider formula using impedances"

Across Variable Divider

(Across Variable $= x = \int v(\lambda) d\lambda$)



By inspection of the diagram

$$\frac{X_2(s)}{X_{in}(s)} = \frac{X_2(s)}{X_2(s) + X_1(s)} \tag{1}$$

and

$$F_1(s) = F_2(s) \tag{2}$$

Define the *compliances* $C_1(s)$ and $C_2(s)$ as

$$C_1(s) = \frac{X_1(s)}{F_1(s)} \qquad C_2(s) = \frac{X_2(s)}{F_2(s)} \tag{3}$$

From (1), using (3),

$$\frac{X_2(s)}{X_{in}(s)} = \frac{C_2(s)F_2(s)}{C_2(s)F_2(s) + C_1(s)F_1(s)} \tag{4}$$

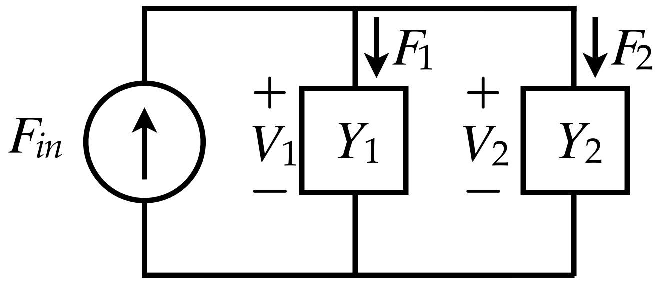
From (4), using (2),

$$\boxed{\frac{X_2(s)}{X_{in}(s)} = \frac{C_2(s)}{C_2(s) + C_1(s)}}$$

"across variable divider formula using compliances"

Through Variable Divider

(Across Variable = v)



By inspection of the diagram

$$\frac{F_2(s)}{F_{in}(s)} = \frac{F_2(s)}{F_2(s) + F_1(s)} \tag{1}$$

and

$$V_1(s) = V_2(s) \tag{2}$$

Define the *admittances* $Y_1(s)$ and $Y_2(s)$ as

$$Y_1(s) = \frac{F_1(s)}{V_1(s)} \qquad Y_2(s) = \frac{F_2(s)}{V_2(s)} \tag{3}$$

From (1), using (3),

$$\frac{F_2(s)}{F_{in}(s)} = \frac{Y_2(s)V_2(s)}{Y_2(s)V_2(s) + Y_1(s)V_1(s)} \tag{4}$$

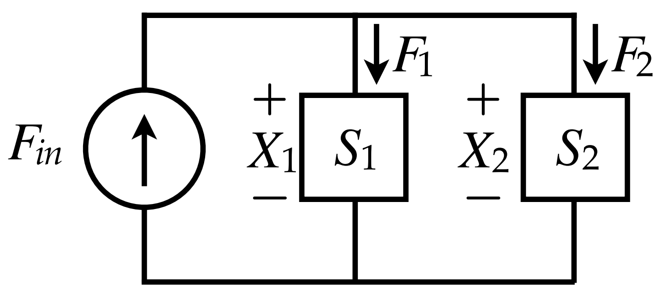
From (4), using (2),

$$\boxed{\frac{F_2(s)}{F_{in}(s)} = \frac{Y_2(s)}{Y_2(s) + Y_1(s)}}$$

"through variable divider formula using admittances"

Through Variable Divider

(Across Variable $= x = \int v(\lambda)d\lambda$)



By inspection of the diagram

$$\frac{F_2(s)}{F_{in}(s)} = \frac{F_2(s)}{F_2(s) + F_1(s)} \tag{1}$$

and

$$X_1(s) = X_2(s) \tag{2}$$

Define the *stiffnesses* $S_1(s)$ and $S_2(s)$ as

$$S_1(s) = \frac{F_1(s)}{X_1(s)} \qquad S_2(s) = \frac{F_2(s)}{X_2(s)} \tag{3}$$

From (1), using (3),

$$\frac{F_2(s)}{F_{in}(s)} = \frac{S_2(s)X_2(s)}{S_2(s)X_2(s) + S_1(s)X_1(s)} \tag{4}$$

From (4), using (2),

$$\boxed{\frac{F_2(s)}{F_{in}(s)} = \frac{S_2(s)}{S_2(s) + S_1(s)}}$$

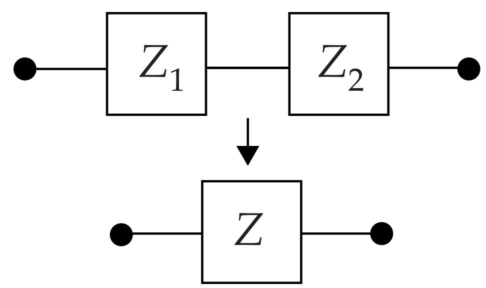
"through variable divider formula using stiffnesses"

Series and parallel combinations of impedances and admittances can be combined. In the following V and F represent the across and through variables respectively of any physical domain.

Series Combination

Elements sharing a common *through* variable are in *series*.

The *impedance* of elements connected in *series* is the sum of the individual *impedances*.

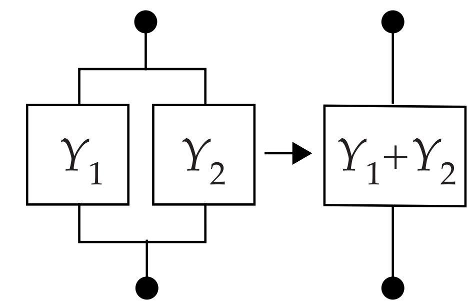


$$Z = Z_1 + Z_2$$

Parallel Combination

Elements sharing a common *across* variable are in *parallel*.

The *admittance* of elements connected in *parallel* is the sum of the individual *admittances*.



$$Y = Y_1 + Y_2$$

$$Z = \frac{1}{Y} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

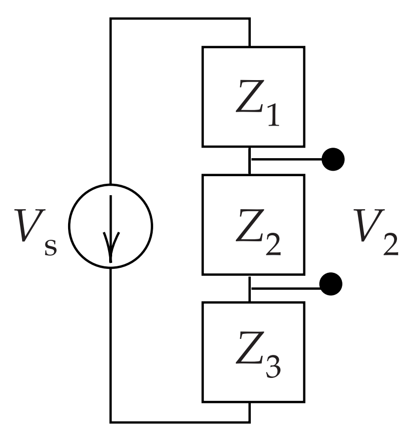
$$Y = \frac{1}{Z} = \frac{1}{Z_1 + Z_2} = \frac{Y_1 Y_2}{Y_1 + Y_2}$$

displacements/rotations: $Z \Rightarrow C$ (compliance) $\Upsilon \Rightarrow S$ (stiffness)

Simple transfer functions can be determined from impedance/admittance properties.

Across Variable Divider

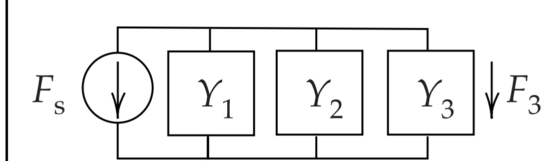
The complex amplitude of the *across variable* across a set of elements in *series* is divided among the elements in proportion their *impedances*.



$$T(s) = \frac{V_2(s)}{V_s(s)} = \frac{Z_2}{Z_1 + Z_2 + Z_3}$$

Through Variable Divider

The complex amplitude of the *through variable* through a set of elements in *parallel* is divided among the elements in proportion their *admittances*.



$$T(s) = \frac{F_2(s)}{F_s(s)} = \frac{Y_2}{Y_1 + Y_2 + Y_3}$$

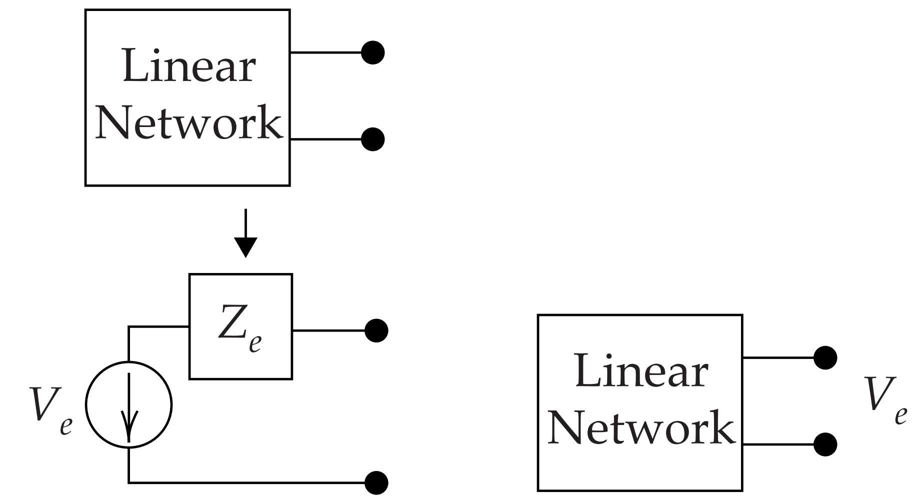
has mathematically identical behavior

Thevenin's Theorem

A linear two-terminal network is equivalent to an across variable source V_e in *series* with an equivalent impedance Z_e, where

Z_e = the impedance of the network with all sources set equal to zero, and

V_e = an *across variable source* equal to the across variable that would appear across the *open* circuit terminals of the network.



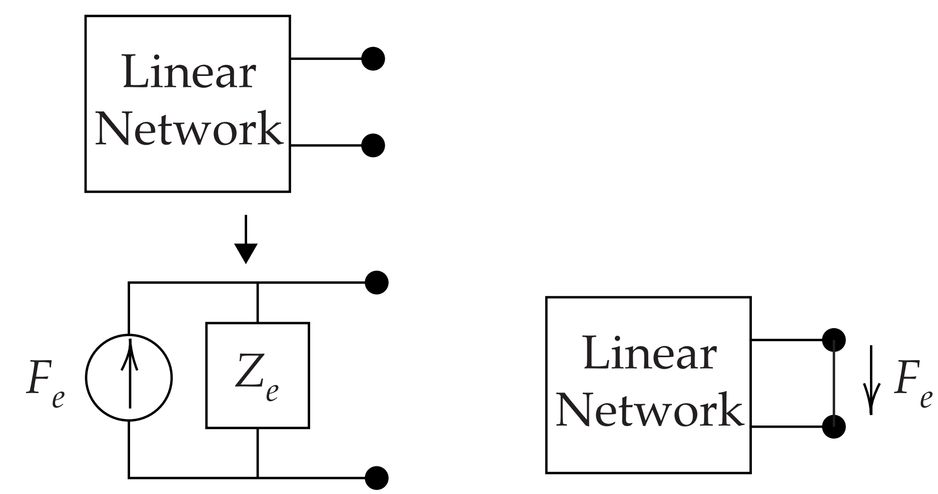
has mathematically identical behavior

Norton's Theorem

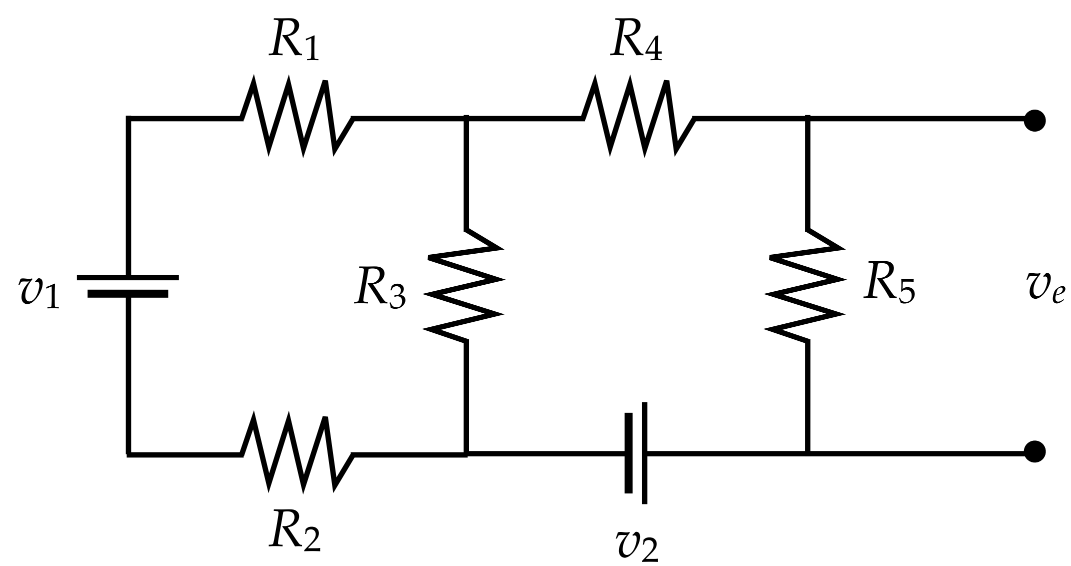
A linear two-terminal network is equivalent to a through variable source F_e in *parallel* with an equivalent impedance Z_e, where

Z_e = the impedance of the network with all sources set equal to zero, and

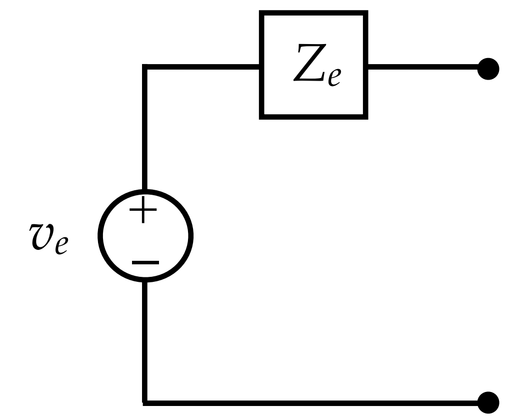
F_e = a *through variable source* equal to the through variable that would flow through the *short* circuited terminals of the network.



Thevenin Equivalent Example



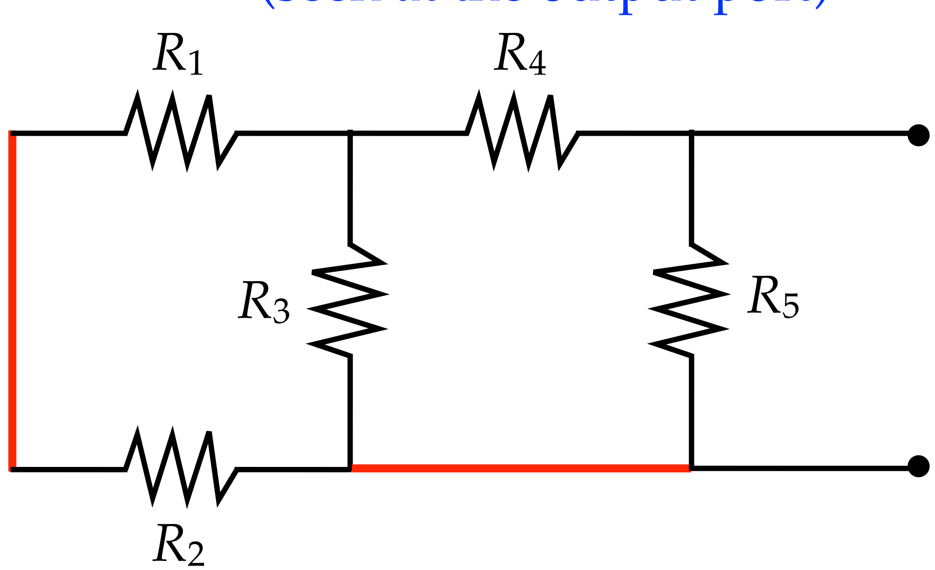
$\Leftrightarrow$



Thevenin Equivalent

$v_e =$ voltage that appears across the open-circuited terminals of the network

$Z_e =$ impedance of the network when the voltage sources are replaced by short circuits (seen at the output port)

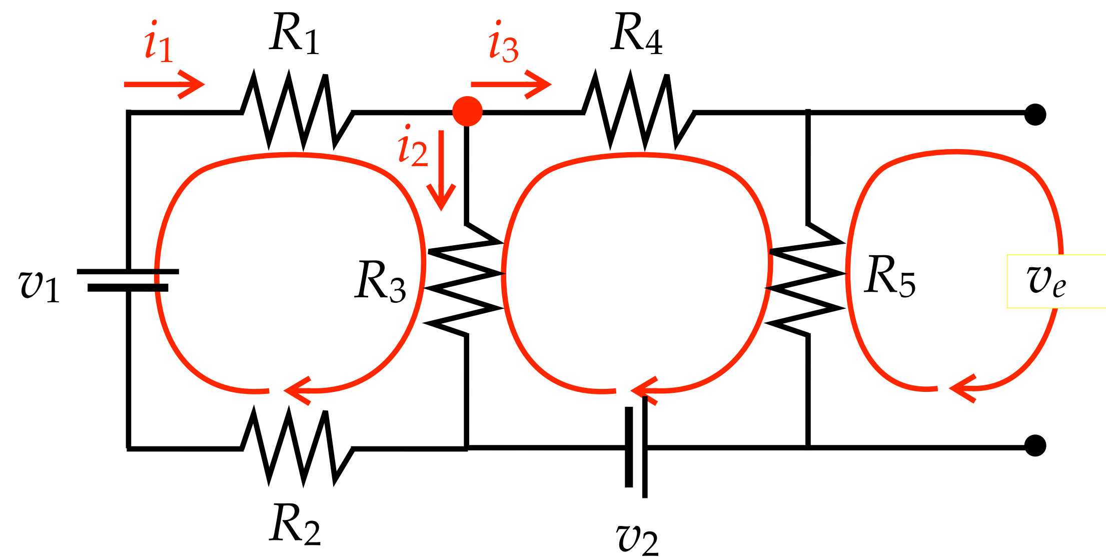


$\Longrightarrow$

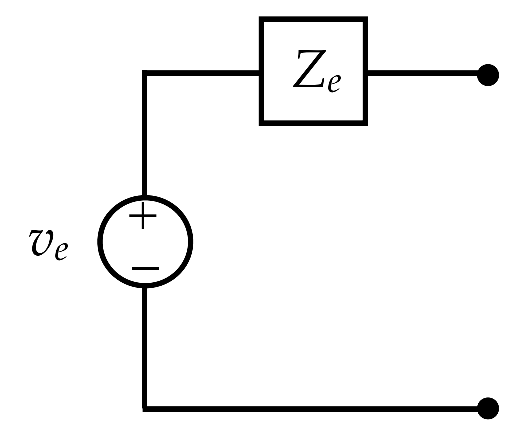
Combine Series & Parallel Elements

$$Z_e = \frac{R_5\left[R_4 + \dfrac{R_3(R_1+R_2)}{(R_3+R_1+R_2)}\right]}{R_5 + \left[R_4 + \dfrac{R_3(R_1+R_2)}{(R_3+R_1+R_2)}\right]}$$

Thevenin Equivalent Example



$\Leftrightarrow$



Thevenin Equivalent

$v_e =$ voltage that appears across the open-circuited terminals of the network

$$\left.\begin{aligned} v_1 - i_1 R_1 - i_2 R_3 - i_1 R_2 &= 0 \\ i_2 R_3 - i_3 R_4 - i_3 R_5 - v_2 &= 0 \\ i_3 R_5 - v_e &= 0 \end{aligned}\right\} \text{Loop Equations}$$

$$i_1 = i_2 + i_3 \qquad \text{Node Equation}$$

4 independent equations in the unknowns i_1, i_2, i_3 and v_e

$$\left.\begin{aligned} v_1 - i_1R_1 - i_2R_3 - i_1R_2 &= 0 \\ i_2R_3 - i_3R_4 - i_3R_5 - v_2 &= 0 \\ i_3R_5 - v_e &= 0 \end{aligned}\right\} \text{ Loop Equations}$$

$$i_1 = i_2 + i_3 \qquad \text{Node Equation}$$

Mathematica solution of simultaneous equations :

In[3]:=

```
Eliminate[{v1 - i1 * R1 - i2 * R3 - i1 * R2 == 0, i2 * R3 - i3 * R4 - i3 * R5 - v2 == 0,
  i3 * R5 - ve == 0, i1 == i2 + i3}, {i1, i2, i3}]
```

Out[3]= $R_1 R_5 v_2 + R_2 R_5 v_2 + R_3 R_5 v_2 + R_1 R_3 v_e + R_2 R_3 v_e + R_1 R_4 v_e + R_2 R_4 v_e + R_3 R_4 v_e + R_1 R_5 v_e + R_2 R_5 v_e + R_3 R_5 v_e == R_3 R_5 v_1$

In[4]:=

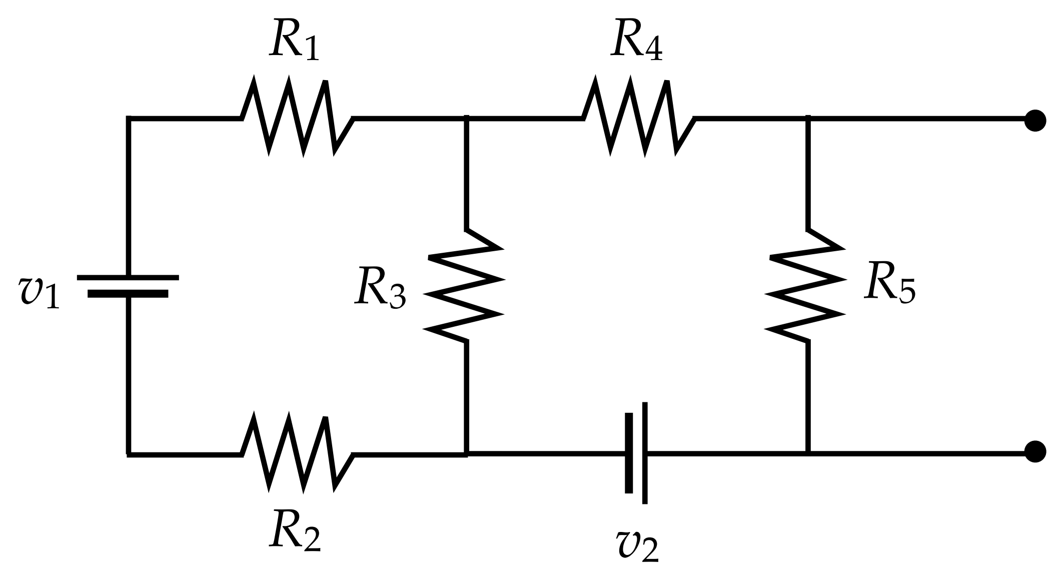
```
Solve[%, ve]
```

Out[4]= $\left\{\left\{v_e \to \frac{R_3 R_5 v_1 - R_1 R_5 v_2 - R_2 R_5 v_2 - R_3 R_5 v_2}{R_1 R_3 + R_2 R_3 + R_1 R_4 + R_2 R_4 + R_3 R_4 + R_1 R_5 + R_2 R_5 + R_3 R_5}\right\}\right\}$

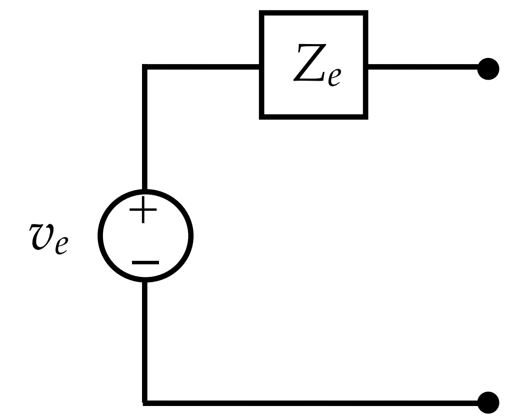
$$\Rightarrow \qquad v_e = \frac{\left[v_1R_3 - v_2(R_1 + R_2 + R_3)\right]R_5}{(R_1 + R_2)R_3 + (R_1 + R_2 + R_3)(R_4 + R_5)}$$

Thevenin Equivalent Example

(Summary)



$\Leftrightarrow$



Thevenin
Equivalent

$v_e =$ voltage that appears across the open-circuited terminals of the network

$Z_e =$ impedance (seen at the output port) of the network when the voltage sources are replaced by short circuits

$$v_e = \frac{\left[v_1 R_3 - v_2(R_1 + R_2 + R_3)\right] R_5}{(R_1 + R_2)R_3 + (R_1 + R_2 + R_3)(R_4 + R_5)}$$

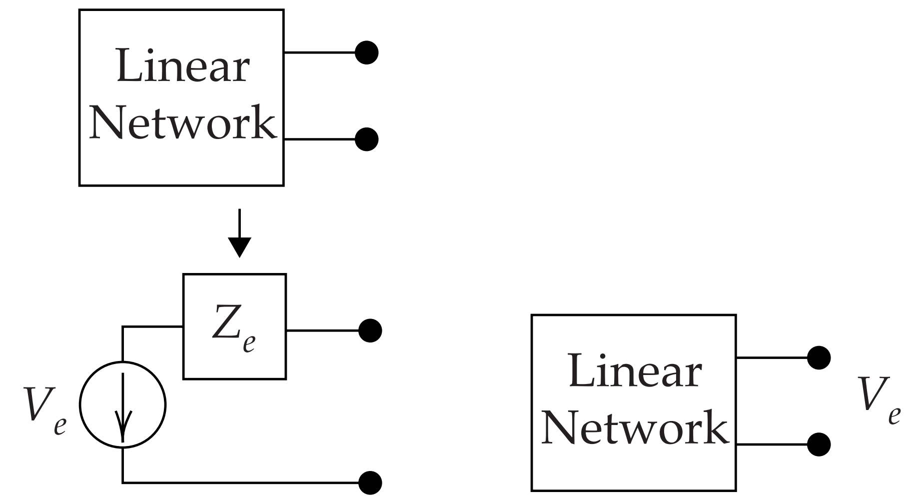
$$Z_e = \frac{R_5\left[R_4 + \dfrac{R_3(R_1 + R_2)}{(R_3 + R_1 + R_2)}\right]}{R_5 + \left[R_4 + \dfrac{R_3(R_1 + R_2)}{(R_3 + R_1 + R_2)}\right]}$$

Thevenin's Theorem

A linear two-terminal network is equivalent to an across variable source V_e in *series* with an equivalent impedance Z_e, where

Z_e = the impedance of the network with all sources set equal to zero, and

V_e = an *across variable source* equal to the across variable that would appear across the *open* circuit terminals of the network.

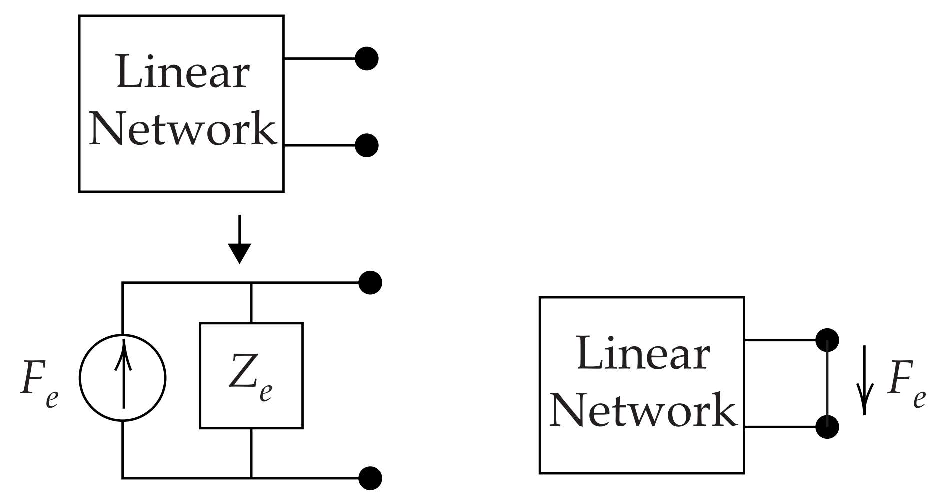


Norton's Theorem

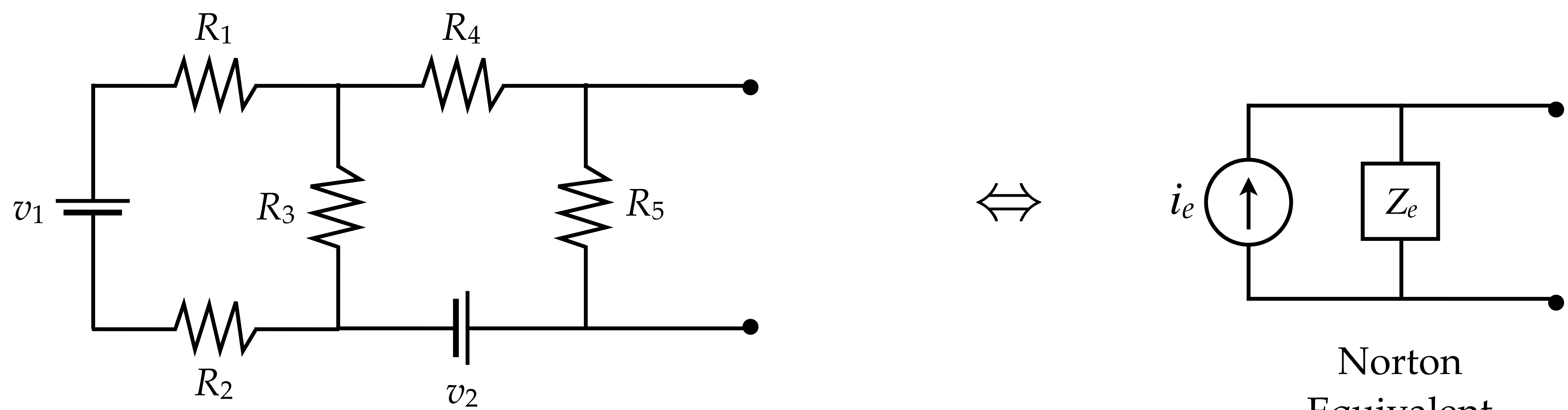
A linear two-terminal network is equivalent to a through variable source F_e in *parallel* with an equivalent impedance Z_e, where

Z_e = the impedance of the network with all sources set equal to zero, and

F_e = a *through variable source* equal to the through variable that would flow through the *short* circuited terminals of the network.



Norton Equivalent Example
(Summary)



Norton Equivalent

$i_e =$ current that would flow through the short-circuited terminals of the network

$Z_e =$ impedance (seen at the output port) of the network when the voltage sources are replaced by short circuits

$$Z_e = \frac{R_5\left[R_4 + \dfrac{R_3(R_1+R_2)}{(R_3+R_1+R_2)}\right]}{R_5 + \left[R_4 + \dfrac{R_3(R_1+R_2)}{(R_3+R_1+R_2)}\right]}$$

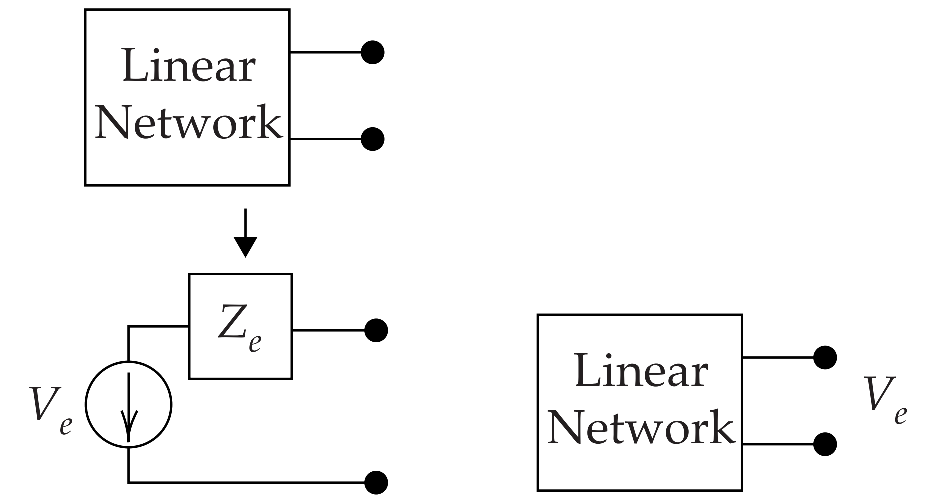
$$i_e = \frac{v_e}{Z_e} = \frac{\left[v_1 R_3 - v_2(R_1+R_2+R_3)\right]R_5}{(R_1+R_2)R_3 + (R_1+R_2+R_3)(R_4+R_5)} \div \frac{R_5\left[R_4 + \dfrac{R_3(R_1+R_2)}{(R_3+R_1+R_2)}\right]}{R_5 + \left[R_4 + \dfrac{R_3(R_1+R_2)}{(R_3+R_1+R_2)}\right]}$$

Thevenin's Theorem

A linear two-terminal network is equivalent to an across variable source V_e in *series* with an equivalent impedance Z_e, where

Z_e = the impedance of the network with all sources set equal to zero, and

V_e = an *across variable source* equal to the across variable that would appear across the *open* circuit terminals of the network.



Norton's Theorem

A linear two-terminal network is equivalent to a through variable source F_e in *parallel* with an equivalent impedance Z_e, where

Z_e = the impedance of the network with all sources set equal to zero, and

F_e = a *through variable source* equal to the through variable that would flow through the *short* circuited terminals of the network.