Analog Filters

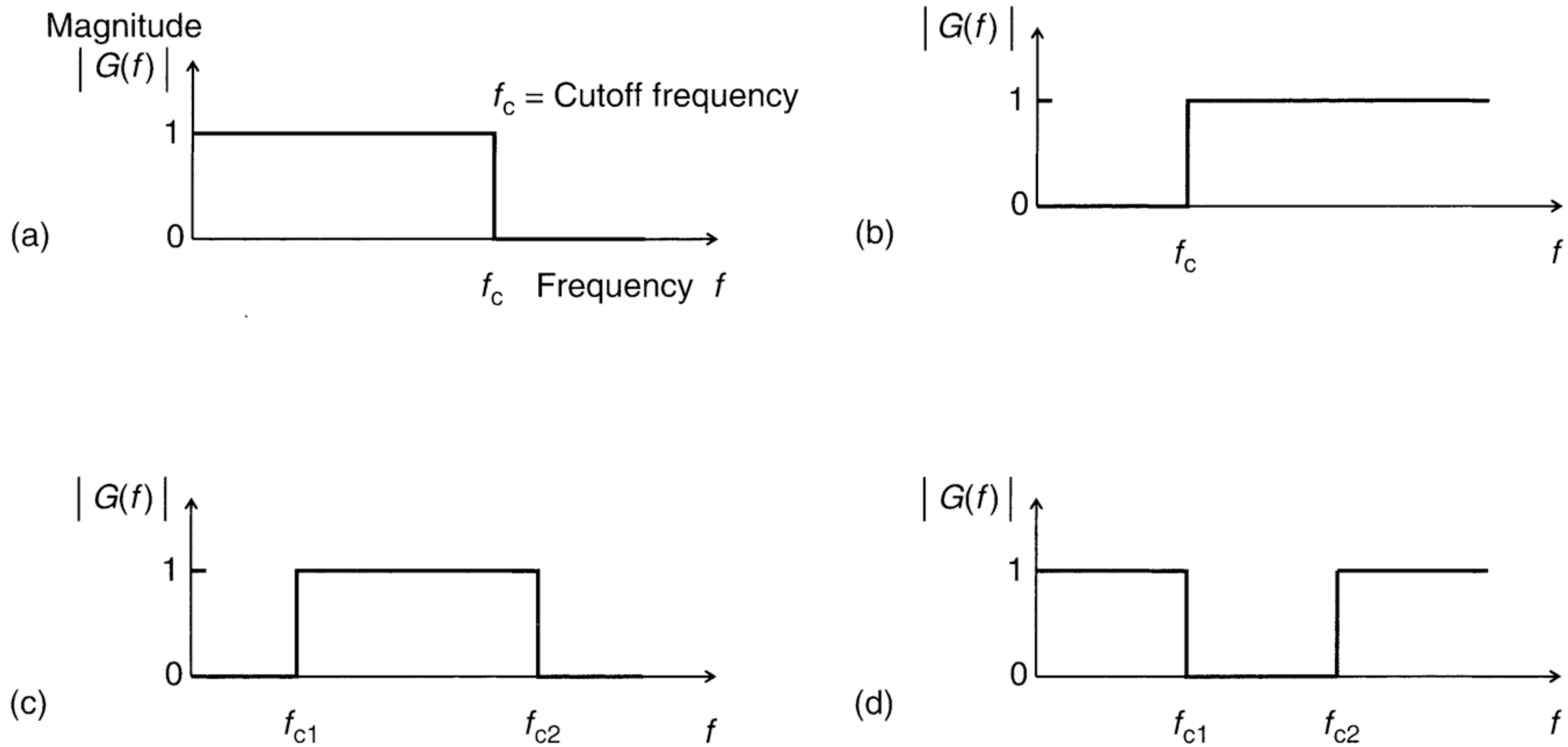
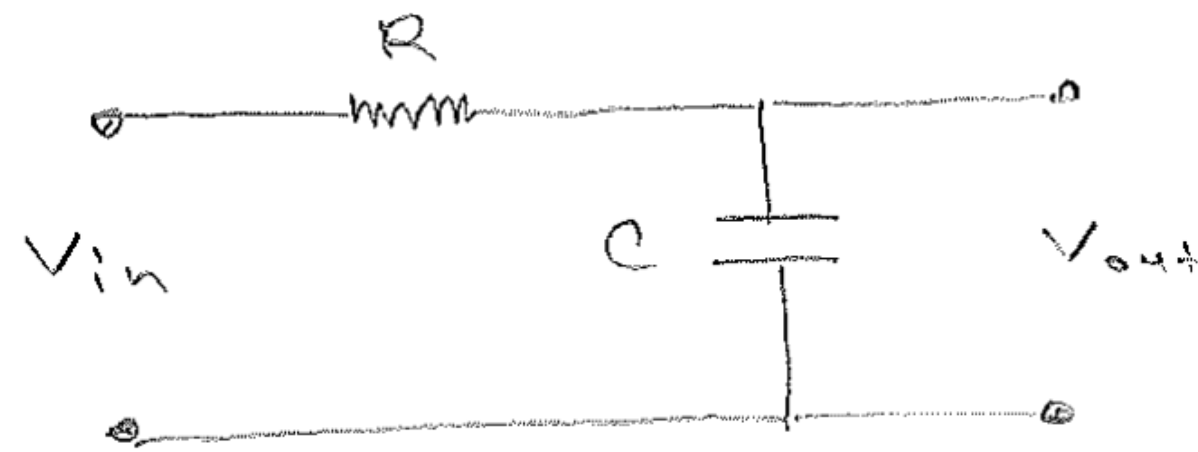


Figure 2.19

Ideal filter characteristics: (a) Low-pass filter. (b) High-pass filter. (c) Band-pass filter. (d) Band-reject (notch) filter.

passive first-order low-pass filter

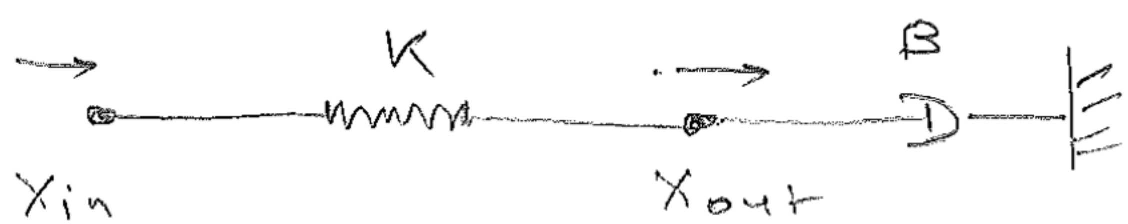


$$V_{out} = \frac{Z_c}{Z_R + Z_c} V_{in} =$$

$$= \frac{1/Cs}{1/Cs + R} V_{in} = \frac{1}{1 + RCs} V_{in}$$

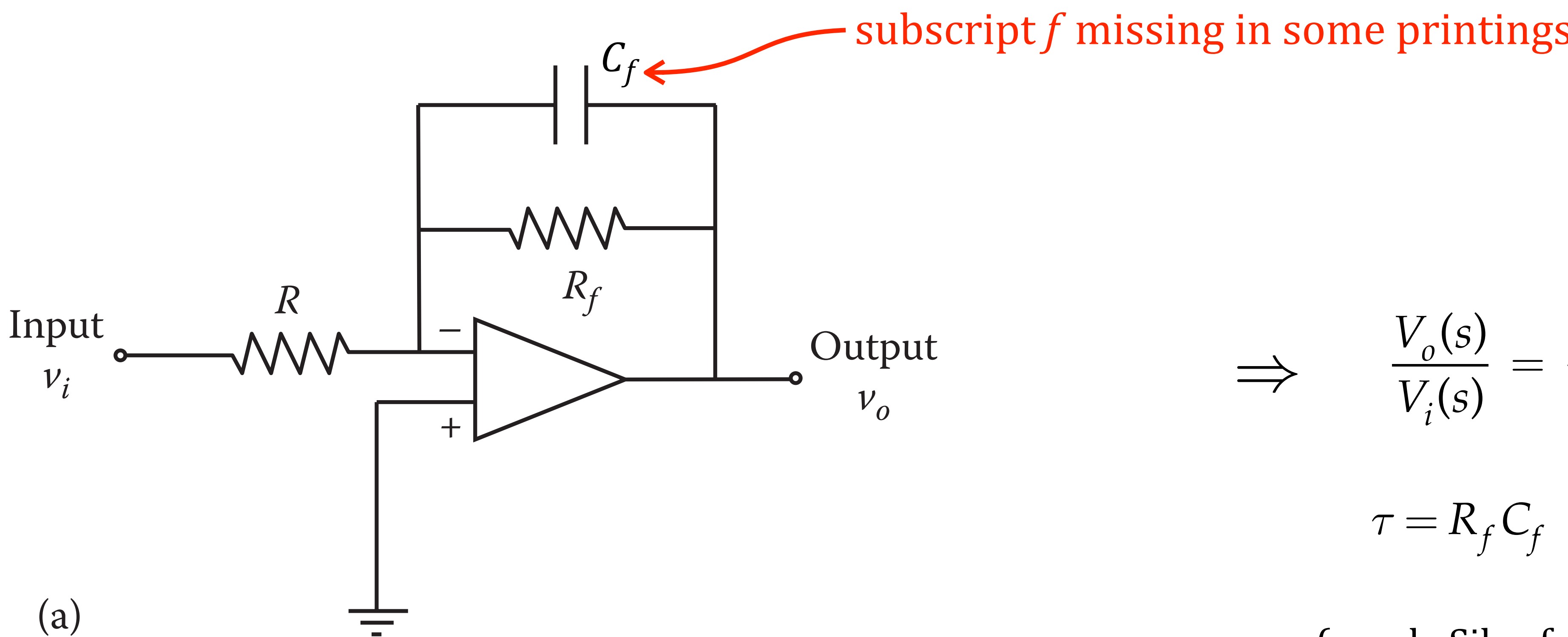
$\tau = RC$ – time constant “RC time constant”

Mechanical analogy



$$X_{out} = \frac{1}{1 + \tau s} X_{in}$$

$$\tau = \frac{B}{K}$$



$$\Rightarrow \quad \frac{V_o(s)}{V_i(s)} = -\frac{k}{\tau s + 1}$$

$$\tau = R_f C_f \qquad k = \frac{R_f}{R}$$

(see de Silva for derivation)

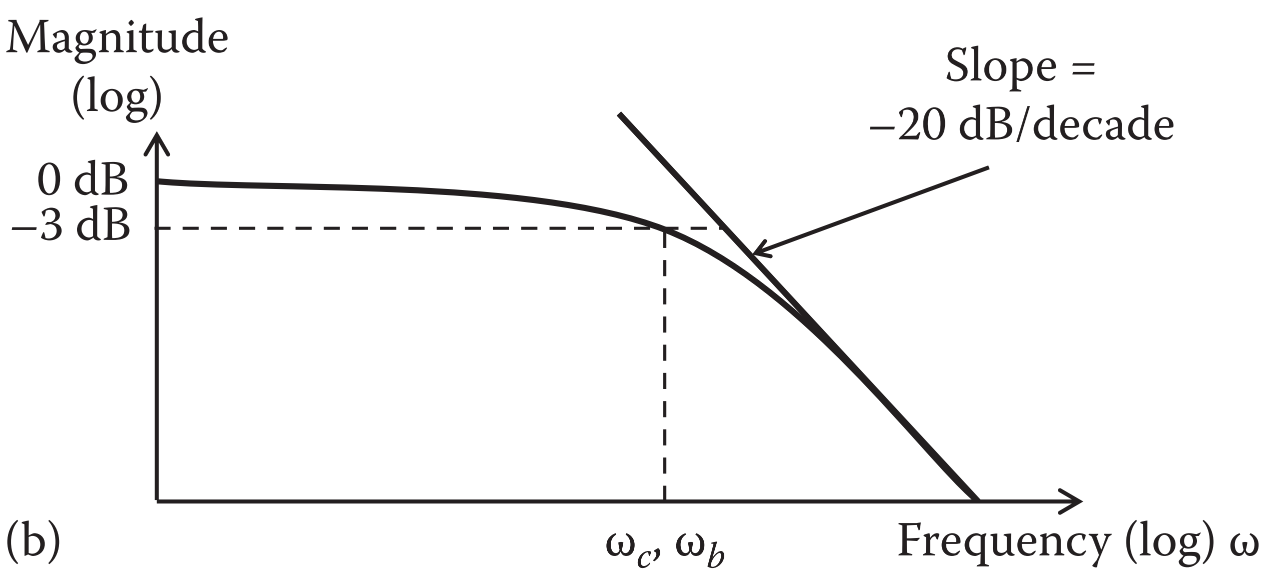
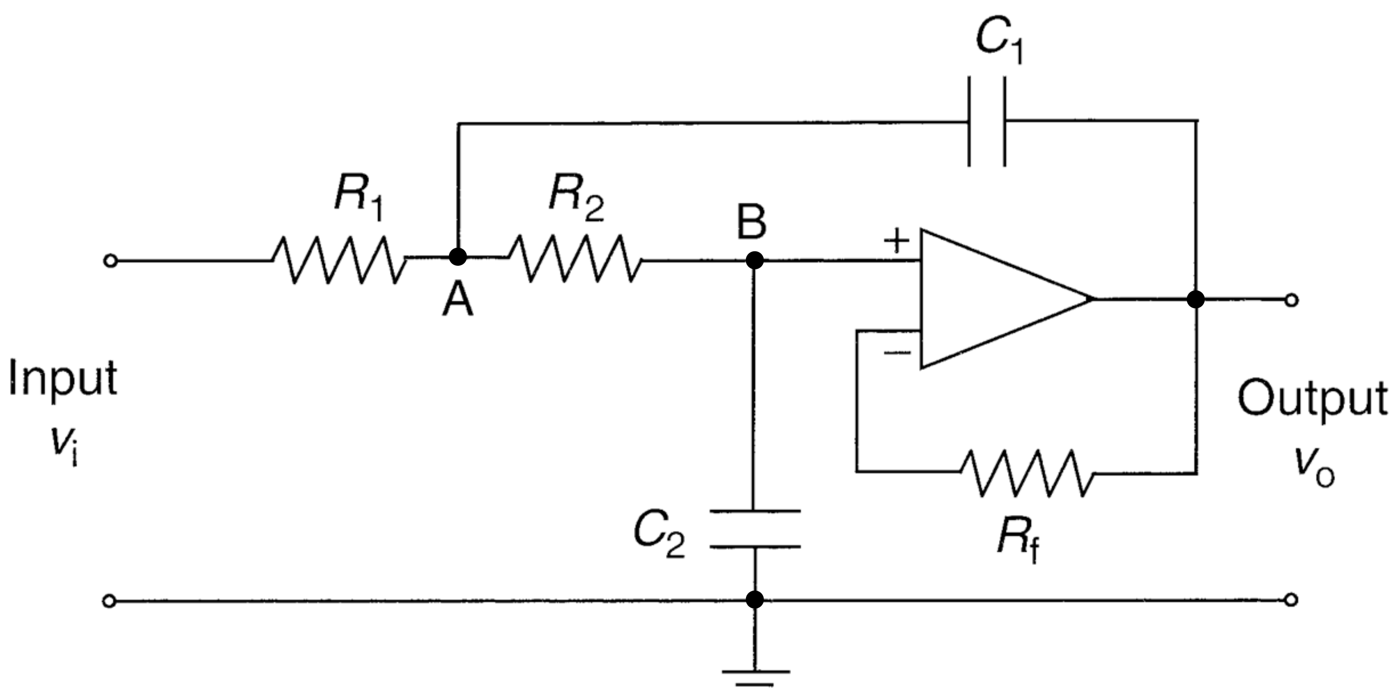


Figure 2.21 (a) A single-pole active low-pass filter (b) the frequency response characteristic.

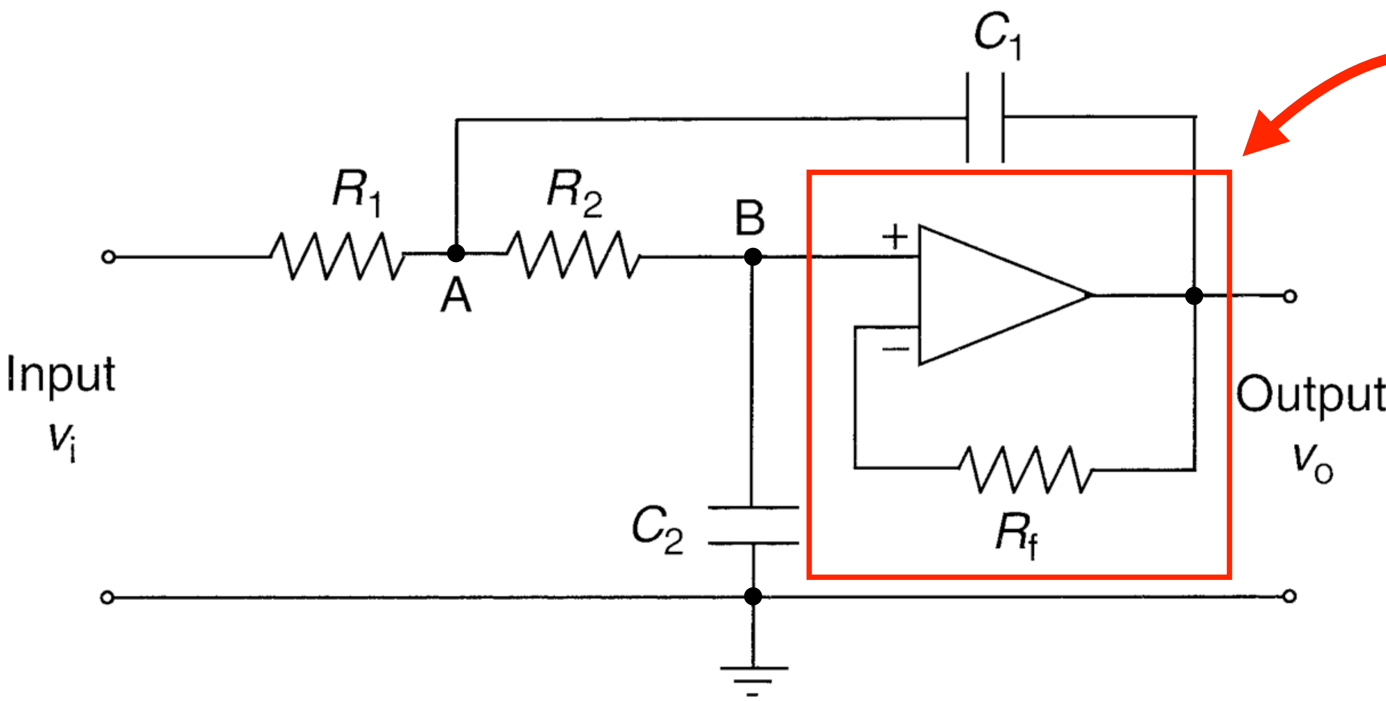


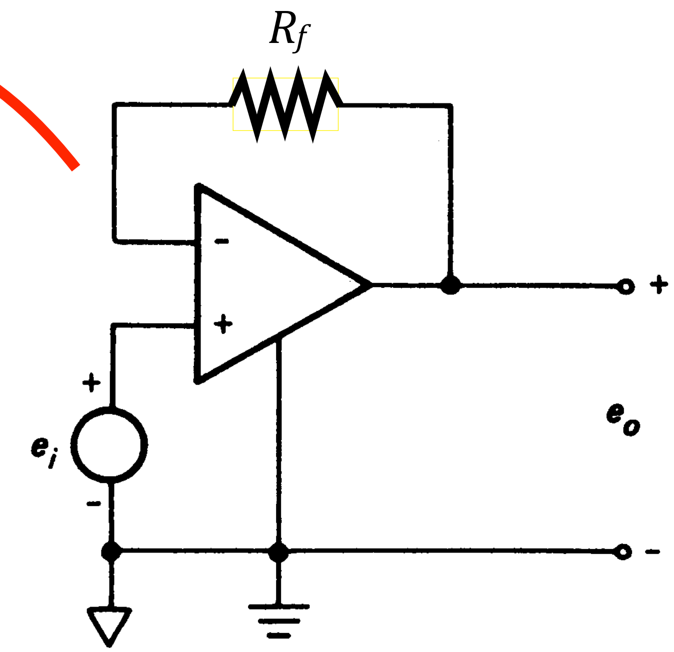
We will show that:

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{\tau_1\tau_2 s^2 + (\tau_2 + \tau_3)s + 1}$$
$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\tau_1 = R_1C_1 \qquad \tau_2 = R_2C_2 \qquad \tau_3 = R_1C_2$$

$$\omega_n = \frac{1}{\sqrt{\tau_1\tau_2}} \qquad \zeta = \frac{\tau_2 + \tau_3}{2\sqrt{\tau_1\tau_2}}$$





Voltage Follower

Node equations for A and B yield:

$$\frac{v_i - v_A}{R_1} + C_1 \frac{d}{dt}(v_o - v_A) = \frac{v_A - v_B}{R_2} = C_2 \frac{dv_B}{dt} \quad (1)$$

Voltage follower yields:

$$v_B = v_o \quad (2)$$

From (1), using (2),

$$\frac{v_i - v_A}{R_1} + C_1 \frac{d}{dt}(v_o - v_A) = \frac{v_A - v_0}{R_2} = C_2 \frac{dv_0}{dt} \quad (3)$$

From the rightmost equation in (3)

$$v_A = R_2 C_2 \frac{dv_0}{dt} + v_0 \quad (4)$$

From the leftmost equation in (3), using (4),

$$\left[v_i - \left(R_2C_2\frac{dv_0}{dt} + v_0\right)\right] + R_1C_1\frac{d}{dt}\left[v_o - \left(R_2C_2\frac{dv_0}{dt} + v_0\right)\right] = R_1\frac{\left(R_2C_2\frac{dv_0}{dt} + v_0\right) - v_0}{R_2} \qquad (5)$$

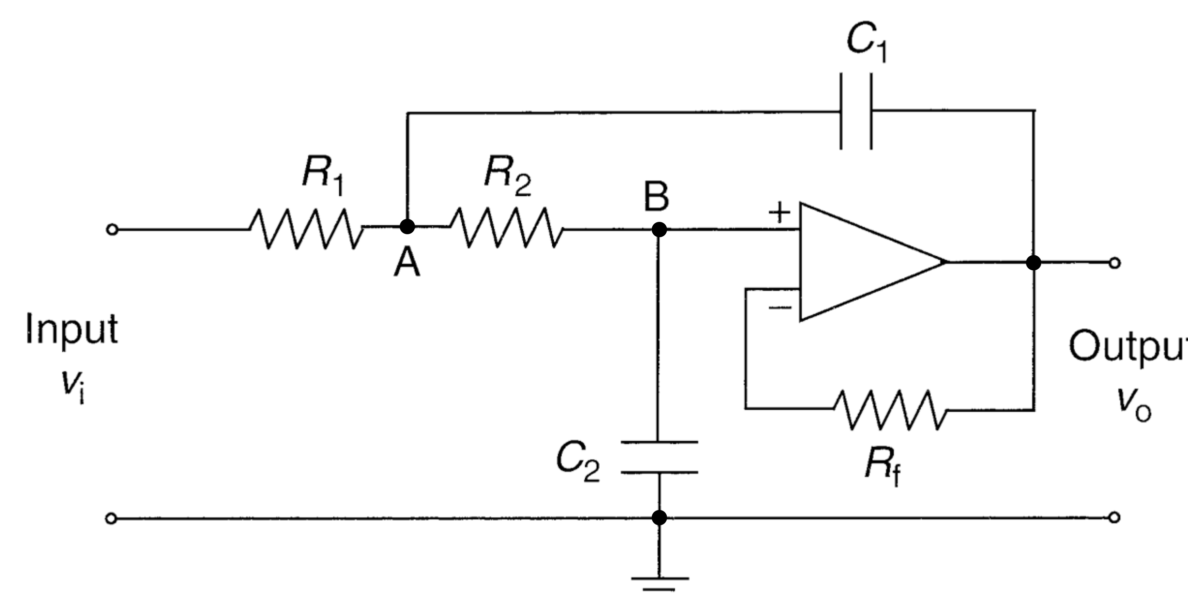
Or

$$v_i = R_1C_2\frac{dv_0}{dt} + R_2C_2\frac{dv_0}{dt} + v_0 + R_1C_1R_2C_2\frac{d^2v_0}{dt^2}$$

Or

$$v_i = \tau_3\frac{dv_0}{dt} + \tau_2\frac{dv_0}{dt} + v_0 + \tau_1\tau_2\frac{d^2v_0}{dt^2}$$

$$\tau_1 = R_1C_1 \qquad \tau_2 = R_2C_2 \qquad \tau_3 = R_1C_2$$



Or

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{\tau_1\tau_2\,s^2 + \left(\tau_2 + \tau_3\right)s + 1}$$

$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \omega_n = \frac{1}{\sqrt{\tau_1\tau_2}} \qquad \zeta = \frac{\tau_2 + \tau_3}{2\sqrt{\tau_1\tau_2}}$$

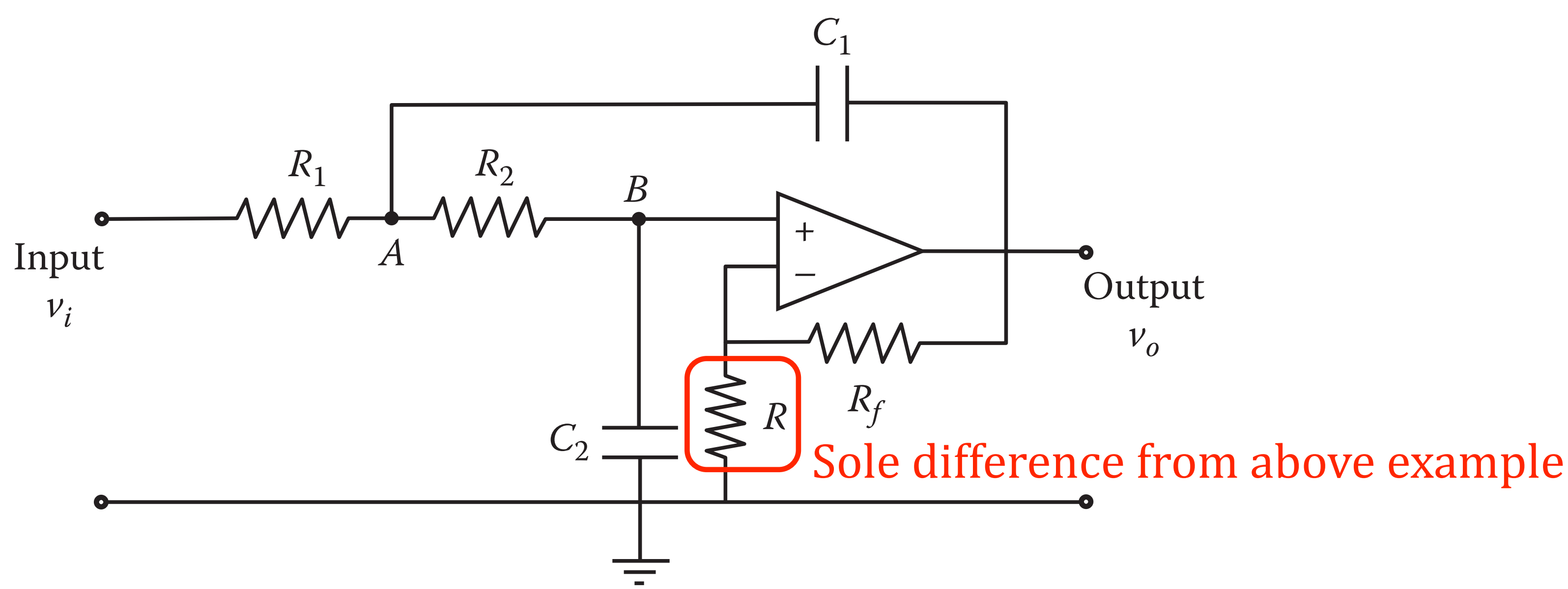


Figure 2.22 A two-pole low-pass Butterworth filter.

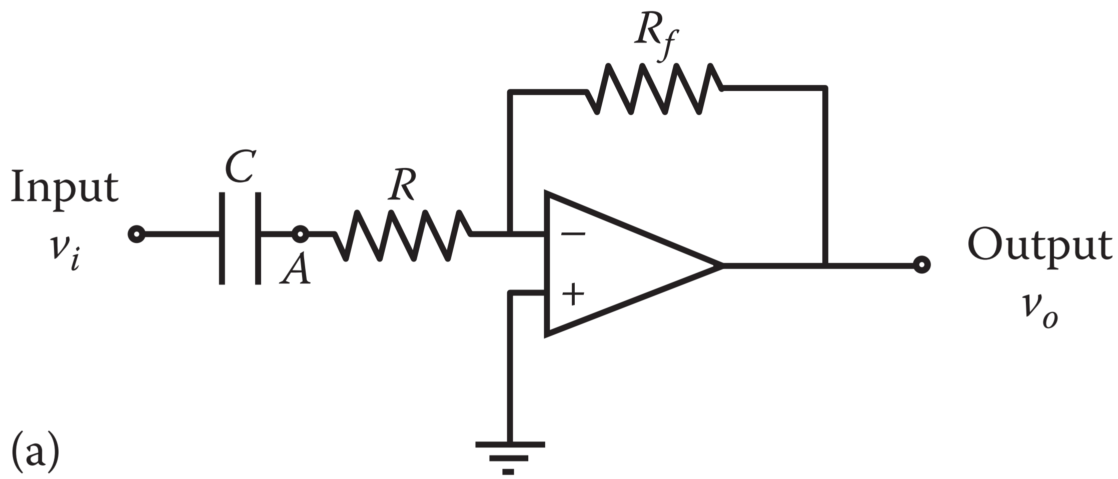
$$\Rightarrow \quad \frac{V_o(s)}{V_i(s)} = \frac{1}{k\left[\tau_1\tau_2 s^2 + \left((1-1/k)\tau_1 + \tau_2 + \tau_3\right)s + 1\right]} = \frac{\omega_n^2}{k\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

$$\tau_1 = R_1 C_1 \qquad \tau_2 = R_2 C_2 \qquad \tau_3 = R_1 C_2 \qquad \omega_n = \frac{1}{\sqrt{\tau_1\tau_2}}$$

$$k = \frac{R}{R_f} \qquad \zeta = \frac{(1-1/k)\tau_1 + \tau_2 + \tau_3}{2\sqrt{\tau_1\tau_2}}$$

(see de Silva for derivation)

de Silva's Eq. 2.54 is incorrect
(sign error)



$$\Rightarrow \quad \frac{V_o(s)}{V_i(s)} = -\frac{R_f}{R}\frac{\tau s}{\tau s+1}$$

$$\tau = RC$$

(derived below)

(error in book: should not be a negative sign here)

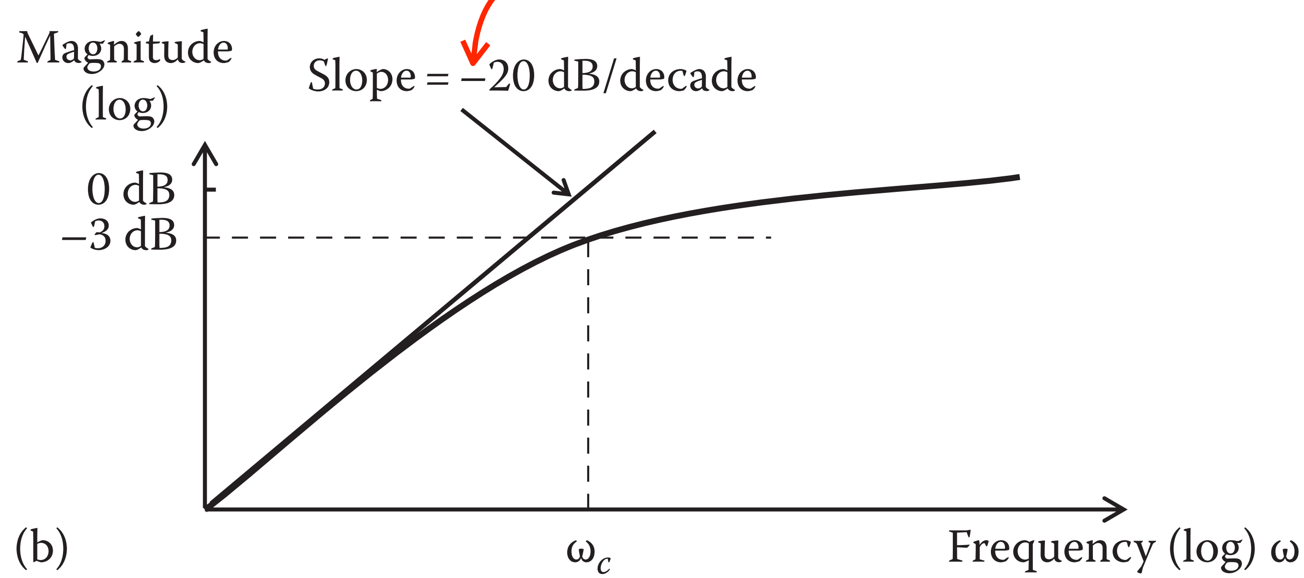
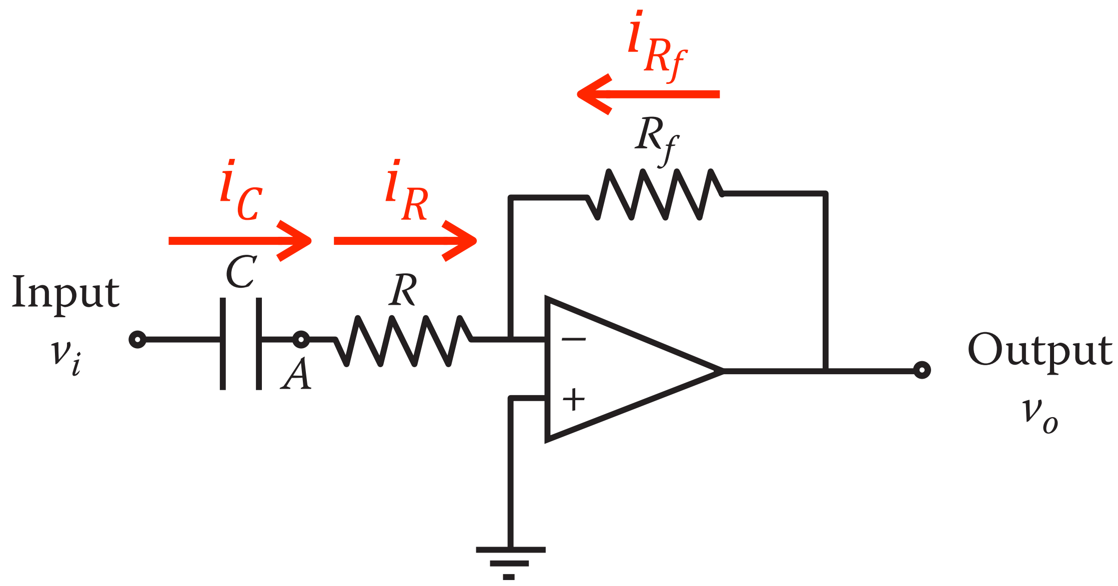


Figure 2.24 (a) A single-pole high-pass filter (b) frequency response characteristic



$$i_C = i_R = -i_{R_f}$$

$$\Rightarrow \quad C\frac{d}{dt}(v_i - v_A) = \frac{v_A}{R} = -\frac{v_o}{R_f}$$

$$\Rightarrow \quad v_A = -\frac{R}{R_f}v_o$$

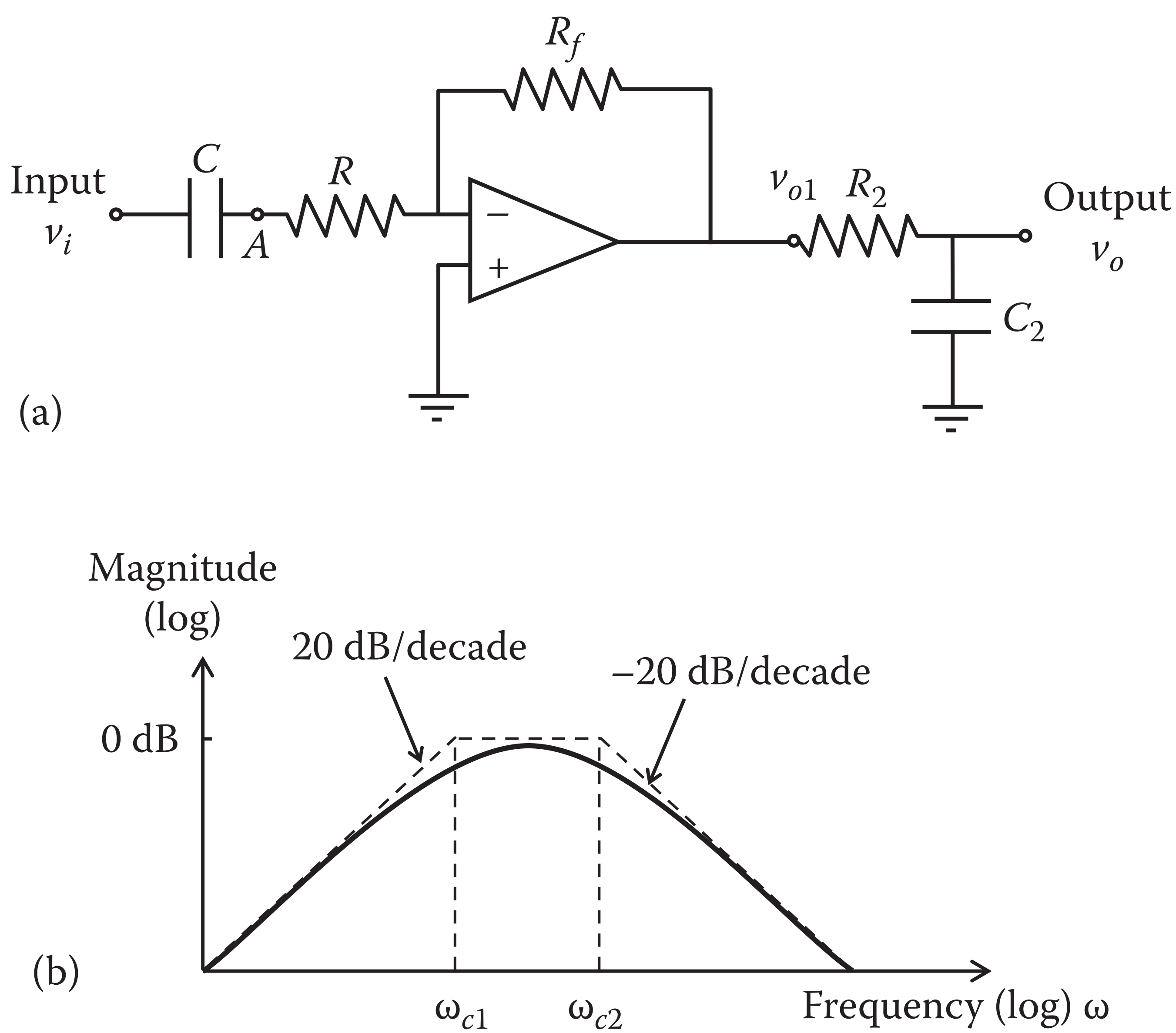
$$\Rightarrow \quad C\frac{d}{dt}\left(v_i + \frac{R}{R_f}v_o\right) = -\frac{v_o}{R_f}$$

$$\Rightarrow \quad C\frac{dv_i}{dt} = -\frac{RC}{R_f}\frac{dv_o}{dt} - \frac{v_o}{R_f}$$

$$\Rightarrow \quad \frac{R_f}{R}RC\frac{dv_i}{dt} = -RC\frac{dv_o}{dt} - v_o$$

$$\Rightarrow \quad \frac{V_o(s)}{V_i(s)} = -\frac{R_f}{R}\frac{\tau s}{\tau s + 1}$$

$$\tau = RC$$



$$\Rightarrow \quad \frac{V_o(s)}{V_i(s)} = \frac{\tau s}{(\tau s + 1)(\tau_2 s + 1)}$$

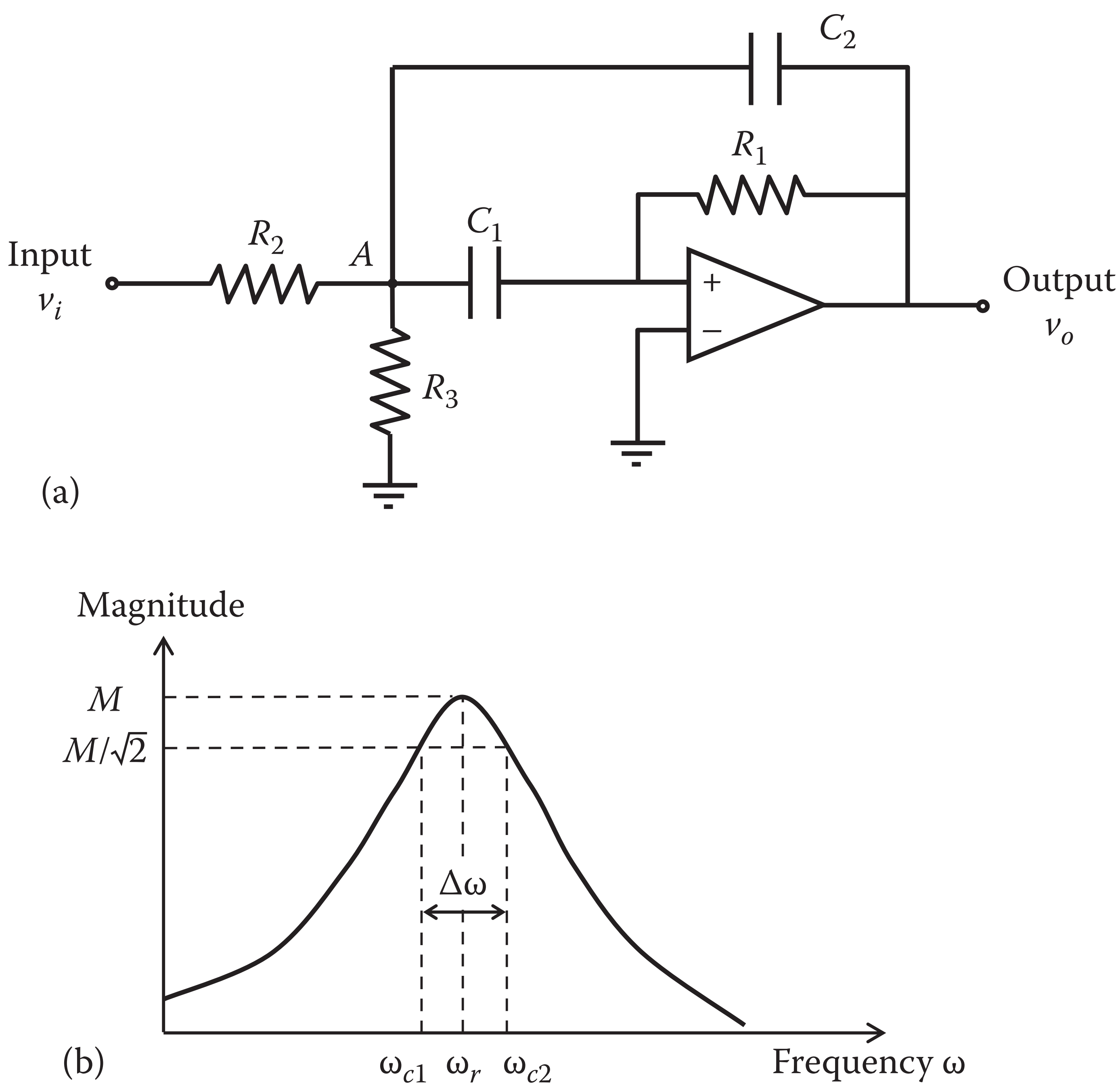
complex roots not possible

$$\tau = RC \qquad \tau_2 = R_2 C_2$$

(see de Silva for derivation)

Figure 2.25 (a) An active band-pass filter

(b) Frequency response characteristic



$$\Rightarrow \quad \frac{V_o(s)}{V_i(s)} = -\frac{\tau_1 s}{\tau_1\tau_2 s^2 + (k_1\tau_1 + \tau_2)s + (1 + k_2)}$$

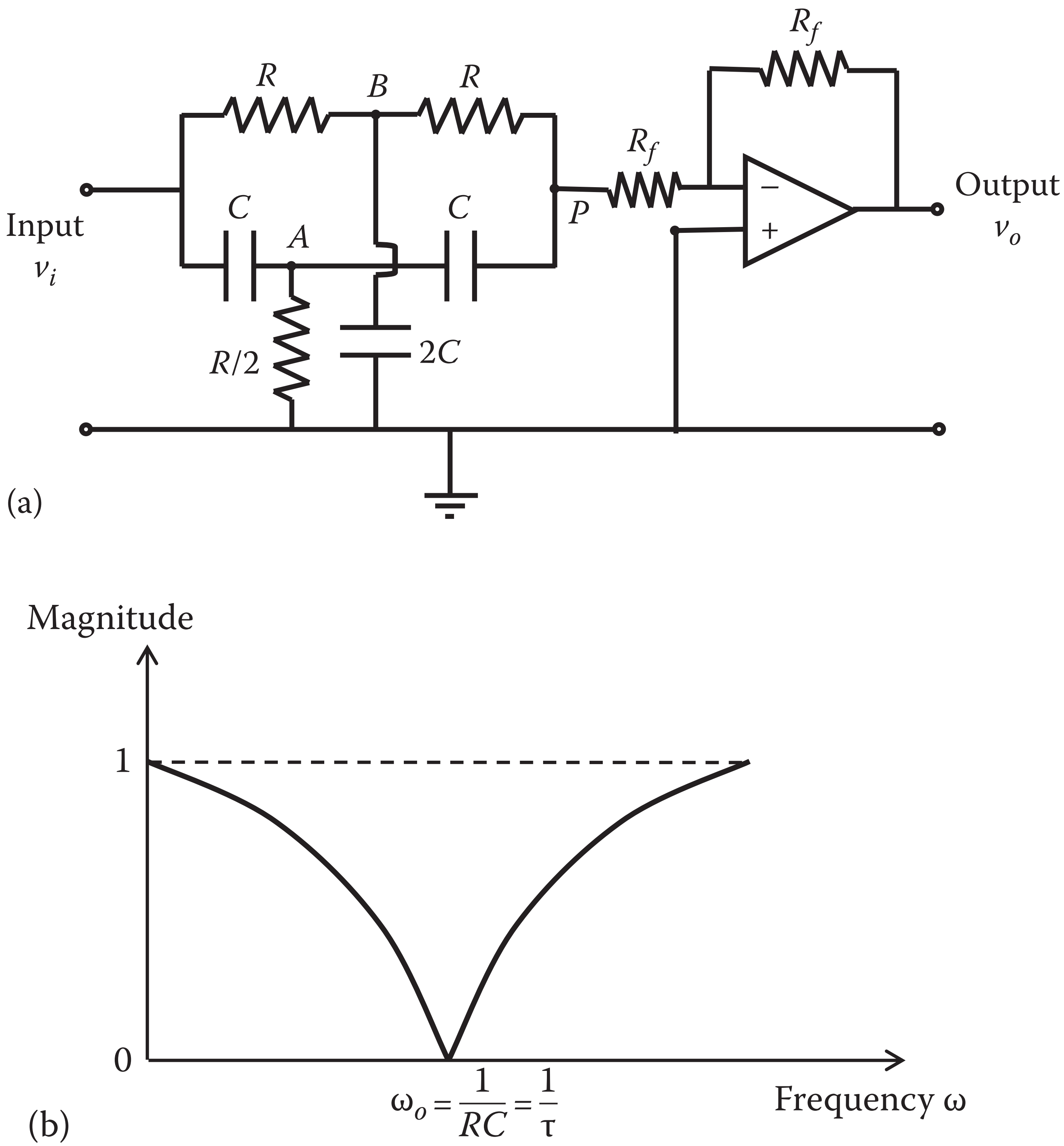
complex roots are possible

$$\tau_1 = R_1C_1 \qquad \tau_2 = R_2C_2$$

$$k_1 = \frac{R_2}{R_1} \qquad k_2 = \frac{R_2}{R_3}$$

(see de Silva for derivation)

Figure 2.26 (a) A resonance-type narrow band-pass filter
(b) frequency response characteristic



$$\Rightarrow \quad \frac{V_o(s)}{V_i(s)} = -\frac{\tau^2 s^2 + 1}{\tau^2 s^2 + (4+k)\tau s + (1+2k)}$$

$$\tau = RC \qquad k = \frac{R}{R_f}$$

(see de Silva for derivation)

a.k.a. a *band-reject* or *notch* filter

Figure 2.27 (a) A twin T filter circuit

(b) Frequency response characteristic

Signal Modulation

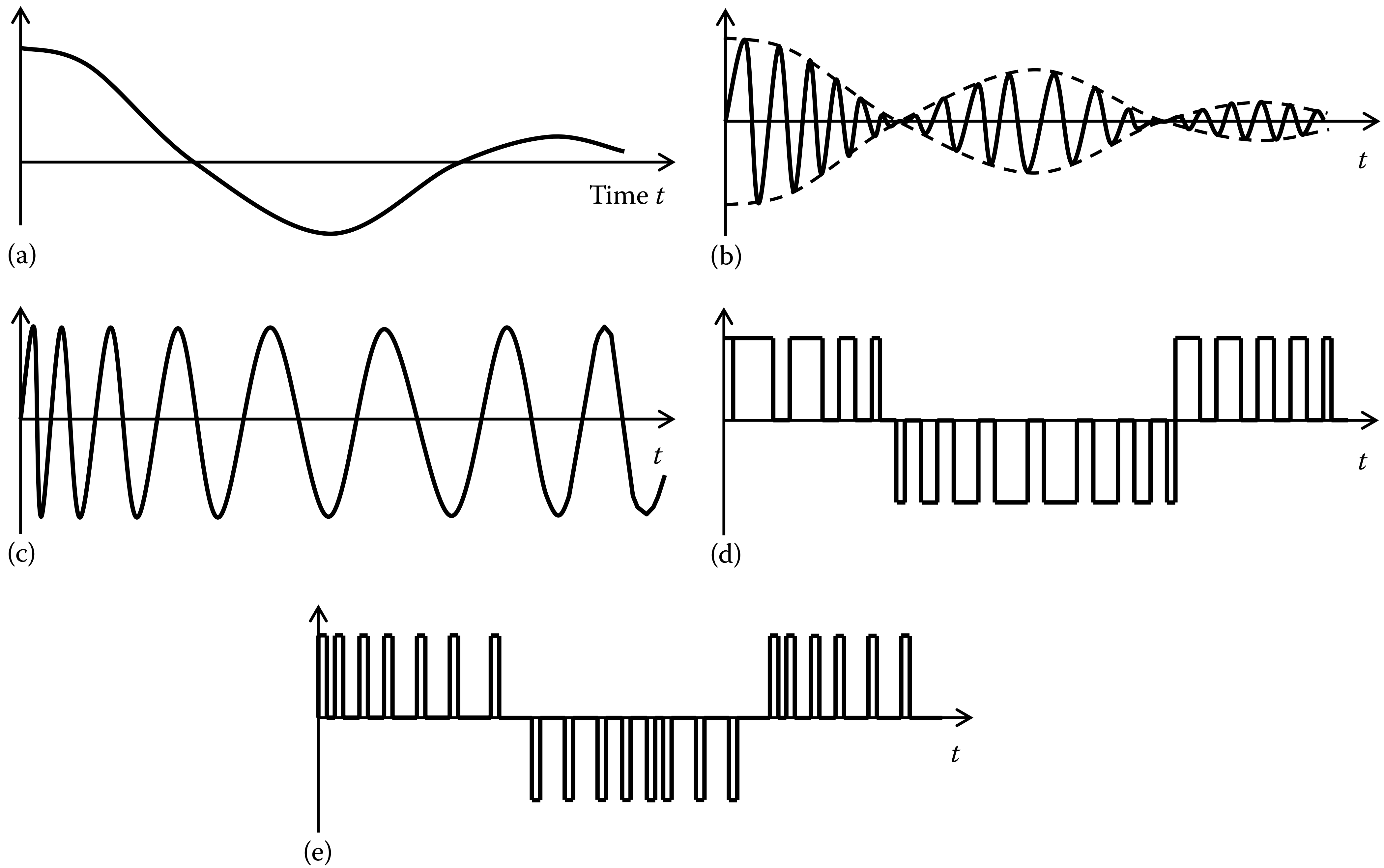
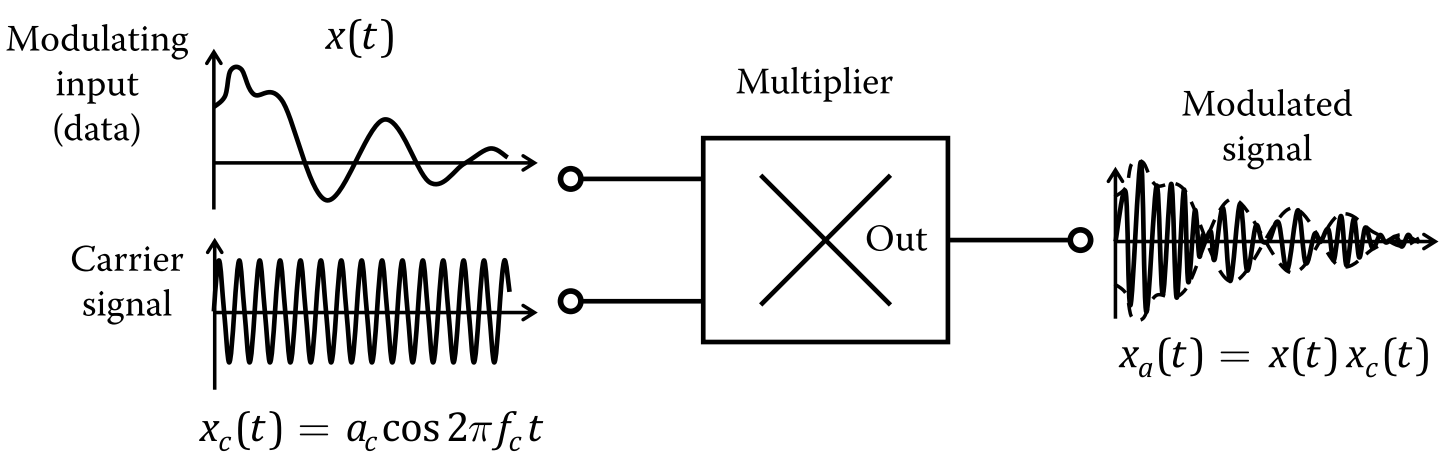


FIGURE 2.28 (a) Modulating signal (data signal), (b) amplitude-modulated (AM) signal, (c) frequency-modulated (FM) signal, (d) pulse-width-modulated (PWM) signal, and (e) pulse-frequency-modulated (PFM) signal.

AM Radio

Amplitude Modulation



Why amplitude modulate radio signals?

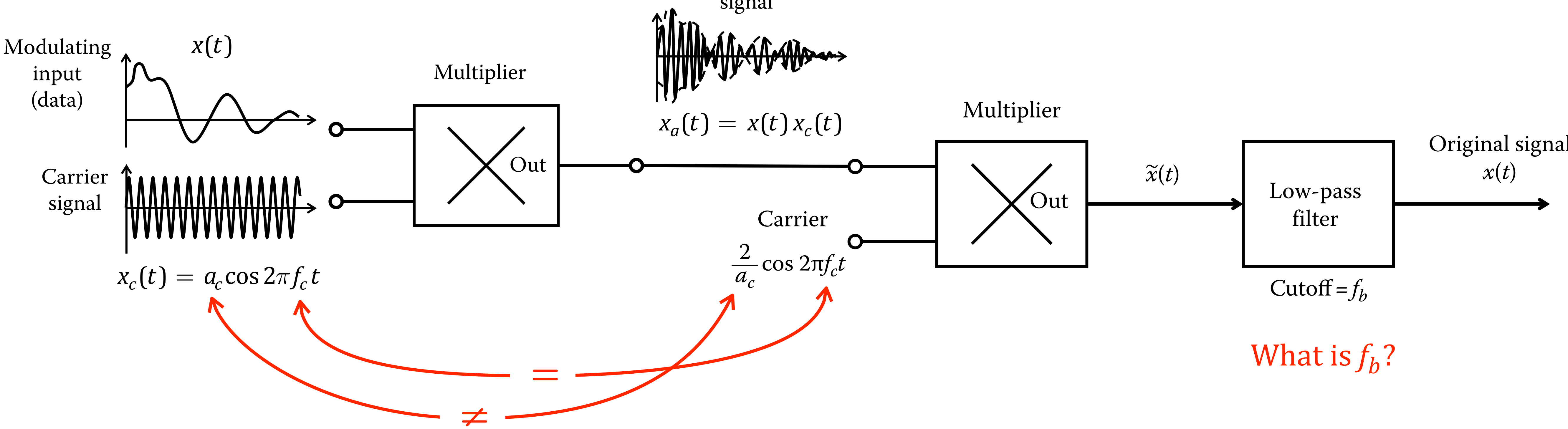
1. The sum of the modulated signals from multiple radio stations can be broadcast over long distances.

2. It is relatively easy to recover one radio station's data signal, $x(t)$, from the sum of the modulated signals from multiple radio stations.

AM Radio

Amplitude Modulation

Demodulation



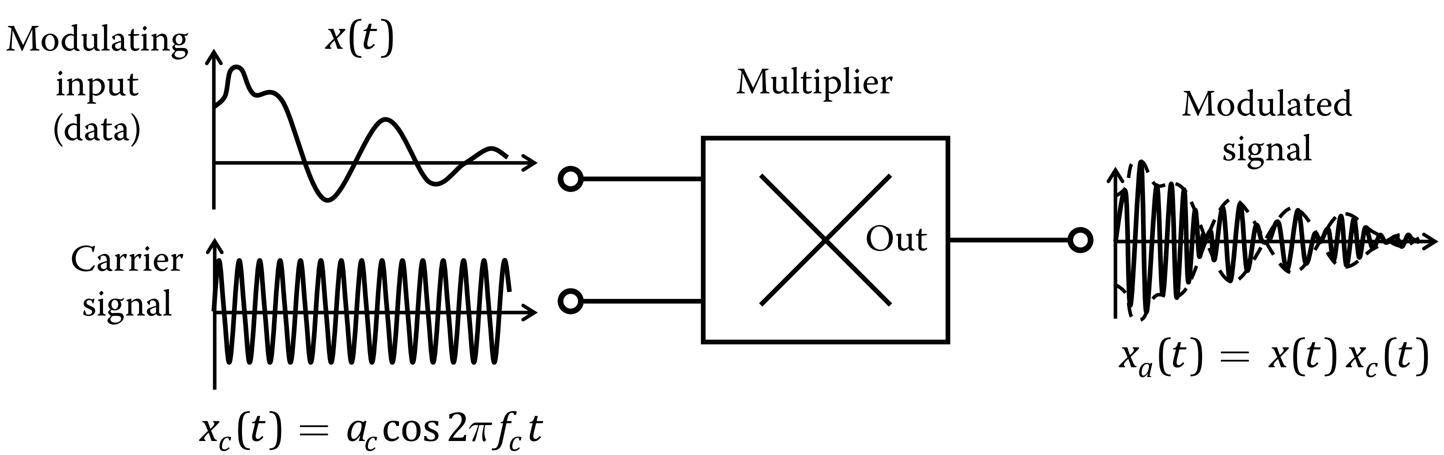
What conditions guarantee that above demodulation scheme recovers $x(t)$ exactly?

How did somebody come up with the above scheme?

What does $x_a(t) = x(t)\, x_c(t)$ look like, in the frequency domain?

AM Radio

Amplitude Modulation



What does $x_a(t) = x(t)\, x_c(t)$ look like, in the frequency domain?

What does $x_a(t) = x(t)\, x_c(t)$ look like, in the frequency domain?

The Fourier transform, $X_a(j\omega)$, of $x_a(t)$, is:

$$X_a(j\omega) = \int_{-\infty}^{+\infty} x_a(t)\, e^{-j\omega t}\, dt$$

where ω is the frequency in radians/sec.

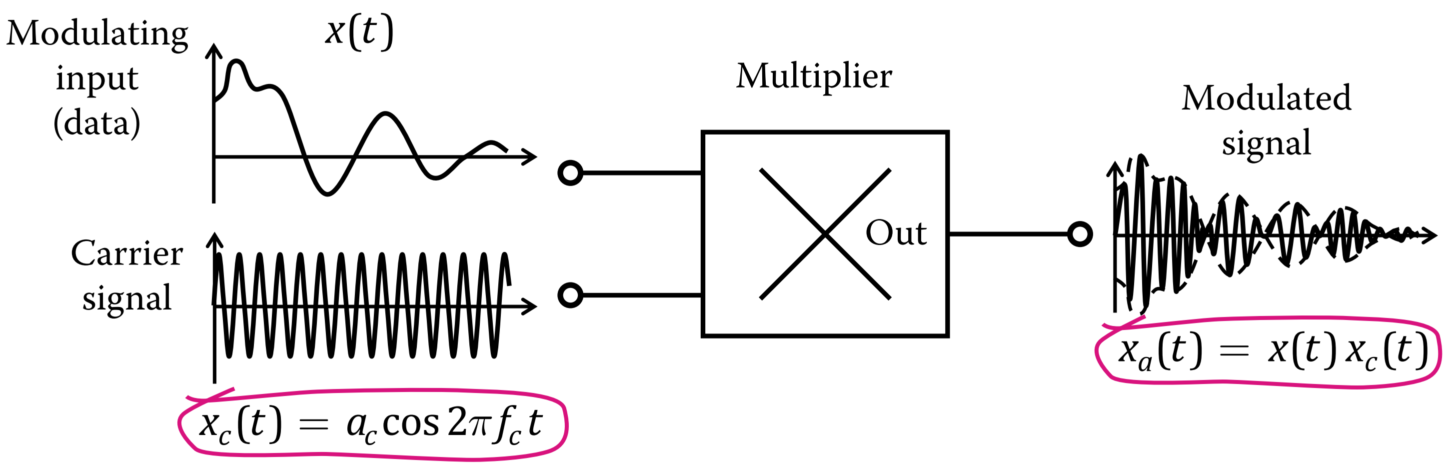
Or we can choose to measure frequency in cycles/sec, and write the above as

$$X_a(f) = \int_{-\infty}^{+\infty} x_a(t)\, e^{-j2\pi f t}\, dt$$

where f is the frequency in cycles/sec.

AM Radio

Amplitude Modulation



What does $x_a(t) = x(t)\, x_c(t)$ look like, in the frequency domain?

The Fourier transform, $X_a(j\omega)$, of $x_a(t)$, is:

$$X_a(j\omega) = \int_{-\infty}^{+\infty} x_a(t)\, e^{-j\omega t}\, dt$$

where ω is the frequency in radians/sec.

Or we can choose to measure frequency in cycles/sec, and write the above as

$$X_a(f) = \int_{-\infty}^{+\infty} x_a(t)\, e^{-j2\pi f t}\, dt$$

where f is the frequency in cycles/sec.

Here

$$x_a(t) = x(t)\, x_c(t) = x(t)\, a_c \cos(2\pi f_c\, t)$$

What does $x_a(t) = x(t)\, x_c(t)$ look like, in the frequency domain?

From Euler's formulas,

$$\cos(\omega_c\, t) = \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}$$

where ω_c is the frequency in radians/sec.

Or we can choose to measure frequency in cycles/sec, and write the above as

$$\boxed{\cos(2\pi f_c t) = \frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}}$$

where f_c is the frequency in cycles/sec.

$$x_a(t) = x(t)\, x_c(t) = x(t)\, a_c \cos(2\pi f_c\, t)$$

What does $x_a(t) = x(t)\, x_c(t)$ look like, in the frequency domain?

The Fourier transform, $X_a(j\omega)$, of $x_a(t)$, is:

$$X_a(j\omega) = \int_{-\infty}^{+\infty} x_a(t)\, e^{-j\omega t}\, dt$$

where ω is the frequency in radians/sec.

Or we can choose to measure frequency in cycles/sec, and write the above as

$$X_a(f) = \int_{-\infty}^{+\infty} x_a(t)\, e^{-j2\pi f t}\, dt$$

where f is the frequency in cycles/sec.

Here

$$x_a(t) = x(t)\, x_c(t) = x(t)\, a_c \cos(2\pi f_c\, t)$$

What does $x_a(t) = x(t)\, x_c(t)$ look like, in the frequency domain?

Now from

$$X_a(f) = \int_{-\infty}^{+\infty} x_a(t)\, e^{-j2\pi f t}\, dt$$

using

$$x_a(t) = x(t)\, x_c(t) = x(t)\, a_c \cos(2\pi f_c t)$$

and

$$\cos(2\pi f_c t) = \frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}$$

we have that

$$X_a(f) = \frac{1}{2} a_c \int_{-\infty}^{+\infty} x(t) \left[e^{j2\pi f_c t} + e^{-j2\pi f_c t} \right] e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2} a_c \int_{-\infty}^{+\infty} x(t) \left[e^{-j2\pi (f - f_c) t} + e^{-j2\pi (f + f_c) t} \right] dt$$

$$= \frac{1}{2} a_c \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f - f_c) t}\, dt \;+\; \frac{1}{2} a_c \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f + f_c) t}\, dt$$

What does $x_a(t) = x(t)\, x_c(t)$ look like, in the frequency domain?

Now from

$$X_a(f) = \int_{-\infty}^{+\infty} x_a(t)\, e^{-j2\pi f t}\, dt$$

using

$$x_a(t) = x(t)\, x_c(t) = x(t)\, a_c \cos(2\pi f_c\, t)$$

and

$$\cos(2\pi f_c t) = \frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}$$

we have that

$$X_a(f) = \frac{1}{2} a_c \int_{-\infty}^{+\infty} x(t) \left[e^{j2\pi f_c t} + e^{-j2\pi f_c t} \right] e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2} a_c \int_{-\infty}^{+\infty} x(t) \left[e^{-j2\pi (f - f_c) t} + e^{-j2\pi (f + f_c) t} \right] dt$$

$$= \frac{1}{2} a_c \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f - f_c) t}\, dt \;+\; \frac{1}{2} a_c \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f + f_c) t}\, dt$$

What does $x_a(t) = x(t)\, x_c(t)$ look like, in the frequency domain?

Finally, using

$$X_a(f) = \frac{1}{2} a_c \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(f-f_c)t}\, dt \;+\; \frac{1}{2} a_c \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(f+f_c)t}\, dt$$

and

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

we have that

$$X_a(f) = \frac{1}{2} a_c \underbrace{X(f - f_c)}_{\substack{\text{Fourier} \\ \text{transfer} \\ \text{of } x(t) \\ \text{shifted } \textit{right} \\ \text{in frequency} \\ \text{by } f_c}} \;+\; \frac{1}{2} a_c \underbrace{X(f + f_c)}_{\substack{\text{Fourier} \\ \text{transfer} \\ \text{of } x(t) \\ \text{shifted } \textit{left} \\ \text{in frequency} \\ \text{by } f_c}}$$

(de Silva's Equation 2.76)

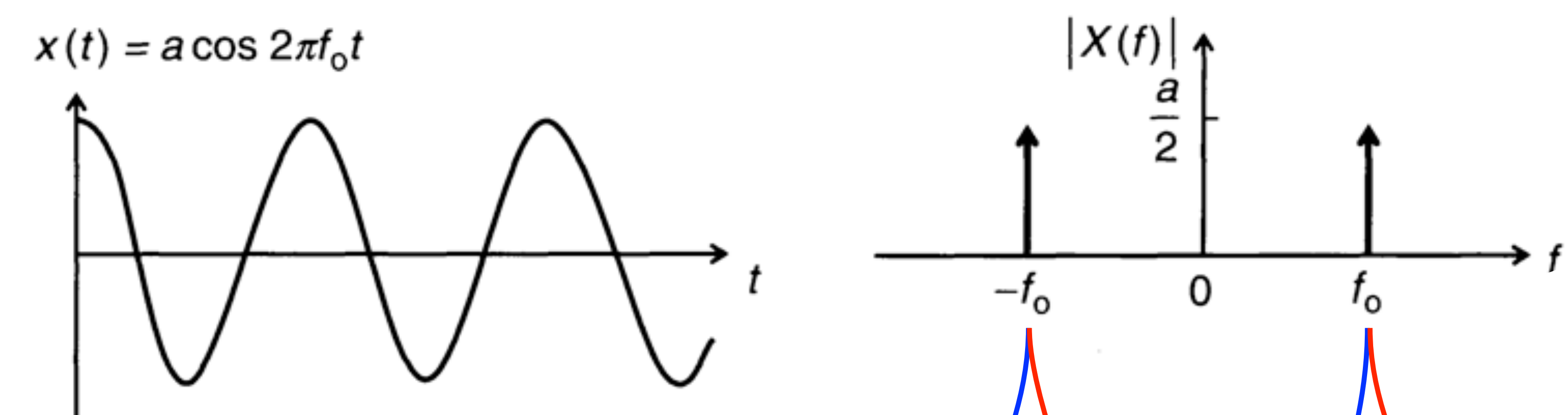
What does $x_a(t) = x(t)\, x_c(t)$ look like, in the frequency domain?

From

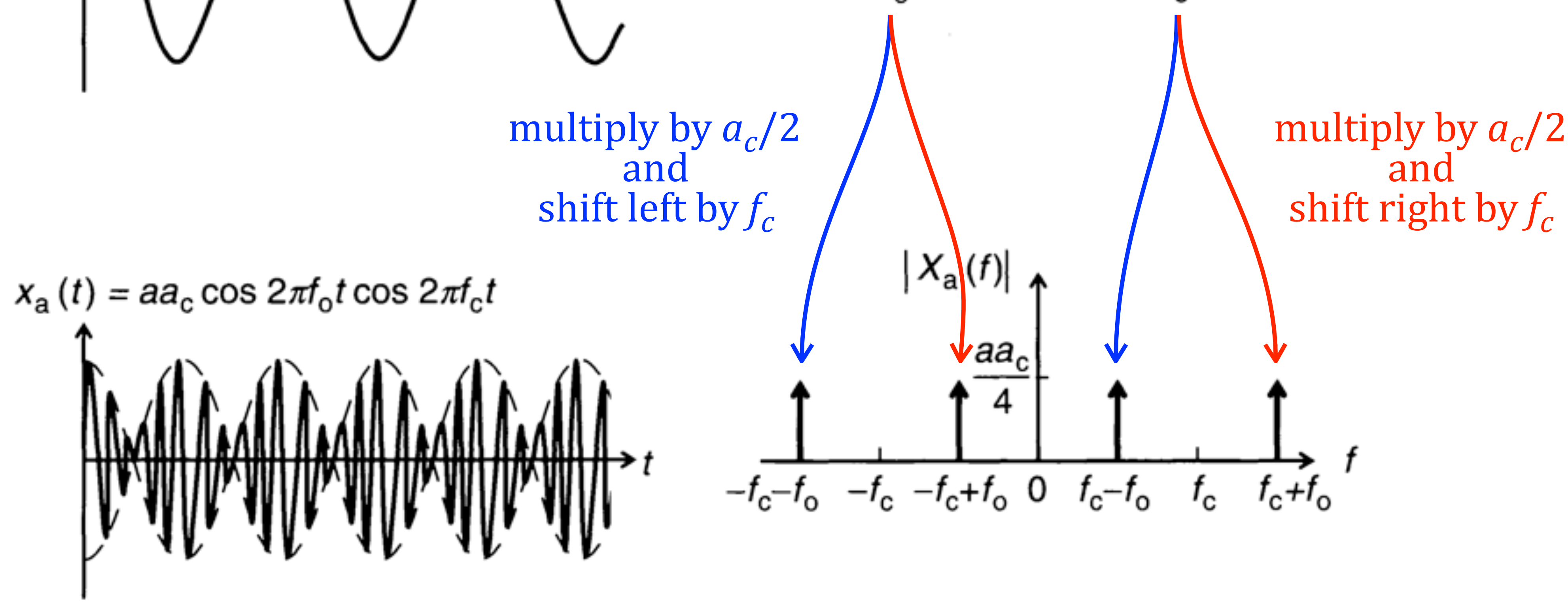
$$X_a(f) = \frac{1}{2} a_c\, X(f - f_c) \;+\; \frac{1}{2} a_c\, X(f + f_c)$$

(de Silva's Equation 2.76)

if



then

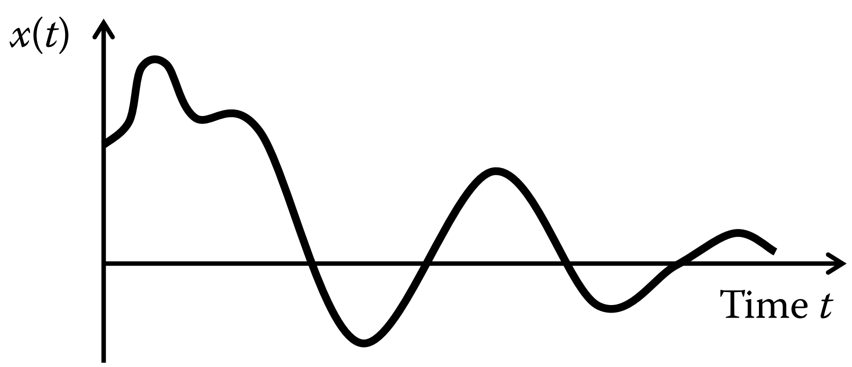


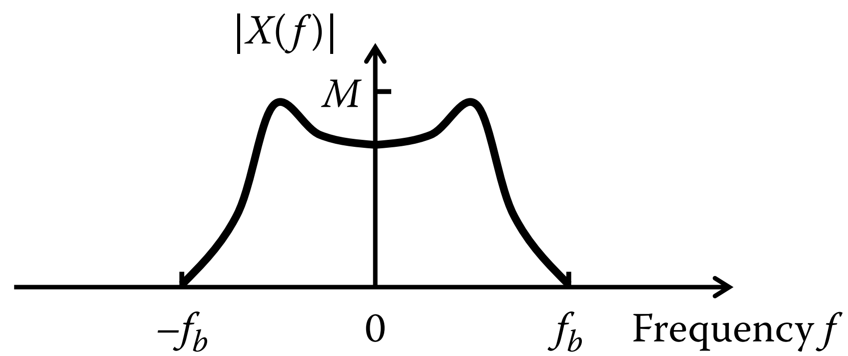
What does $x_a(t) = x(t)\, x_c(t)$ look like, in the frequency domain?

From

$$X_a(f) = \frac{1}{2} a_c\, X(f - f_c) \;+\; \frac{1}{2} a_c\, X(f + f_c) \qquad \text{(de Silva's Equation 2.76)}$$

if

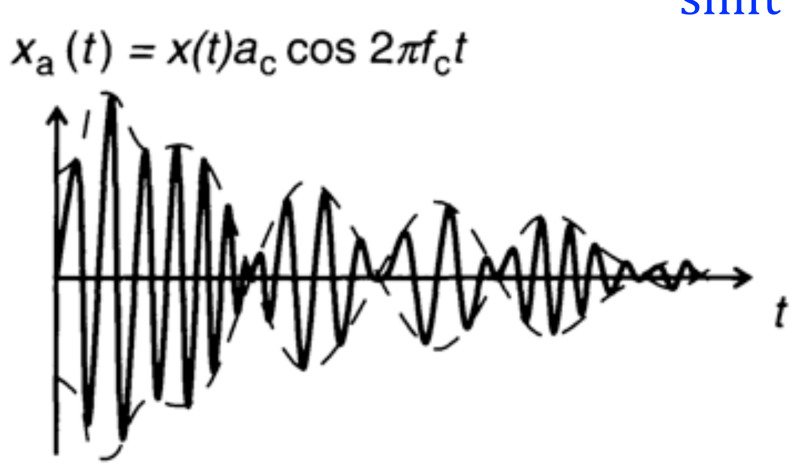


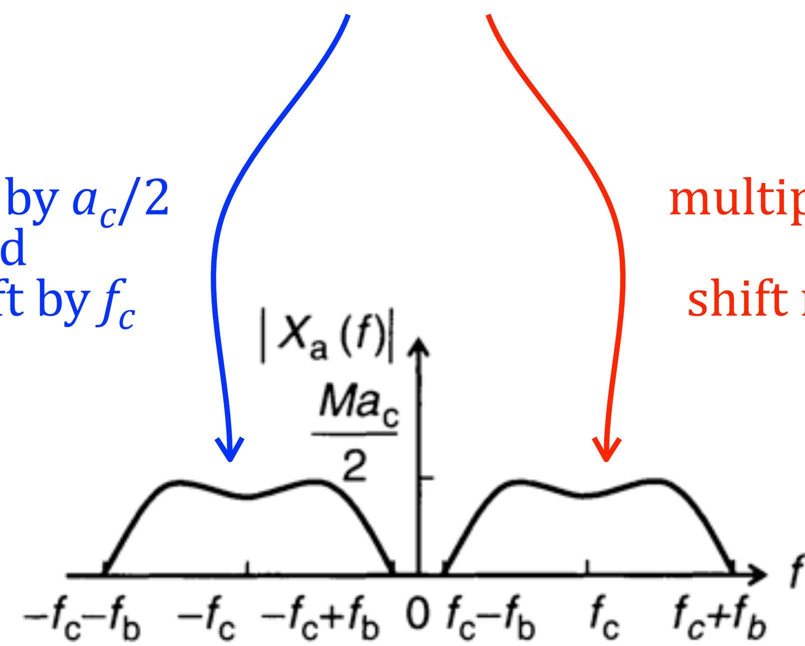


then

multiply by $a_c/2$
and
shift left by f_c

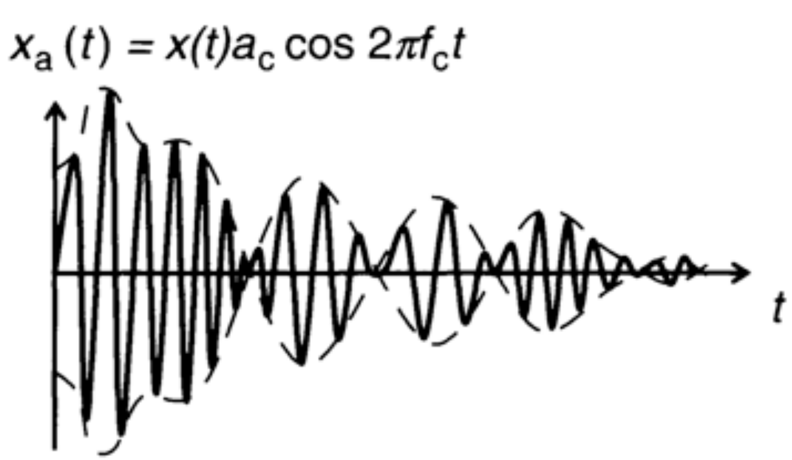
multiply by $a_c/2$
and
shift right by f_c

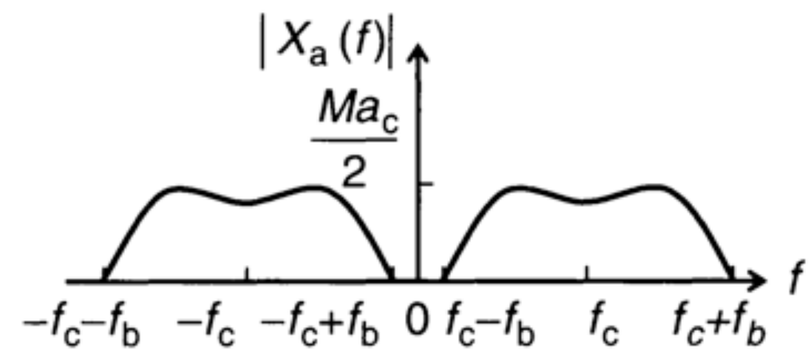




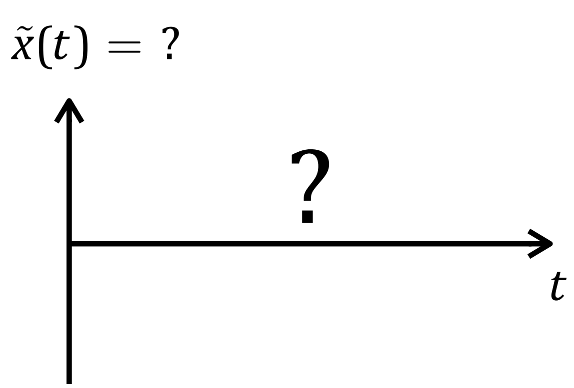
(art work here is very sketchy)

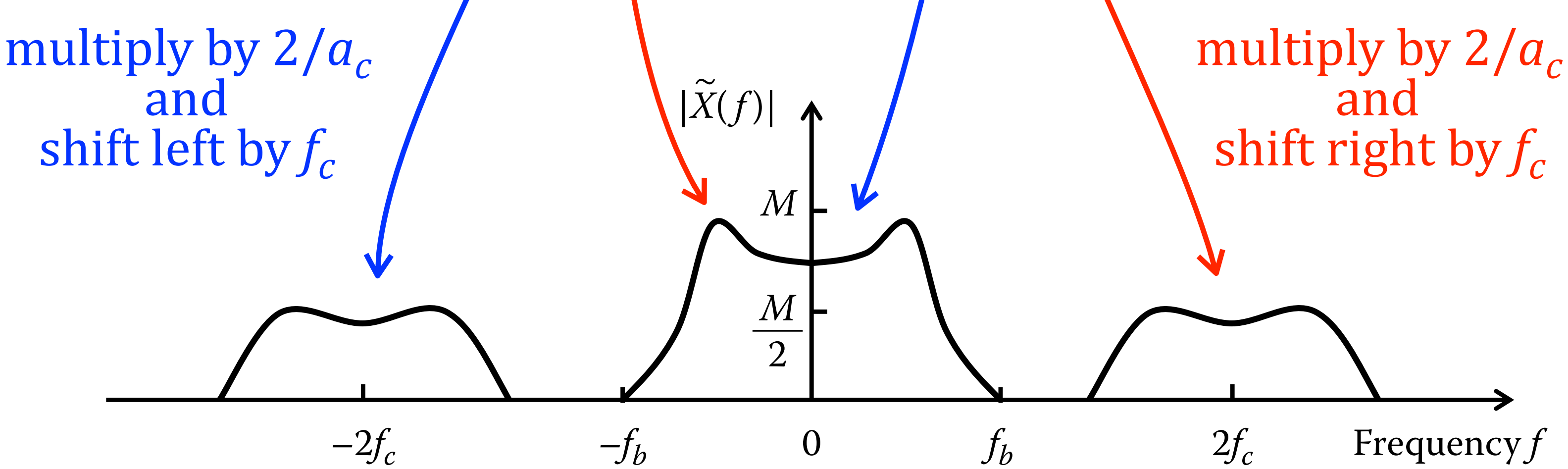
If we take



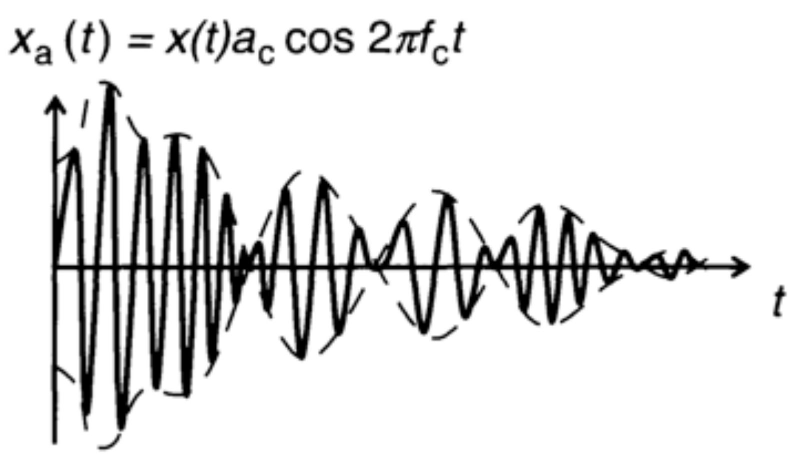


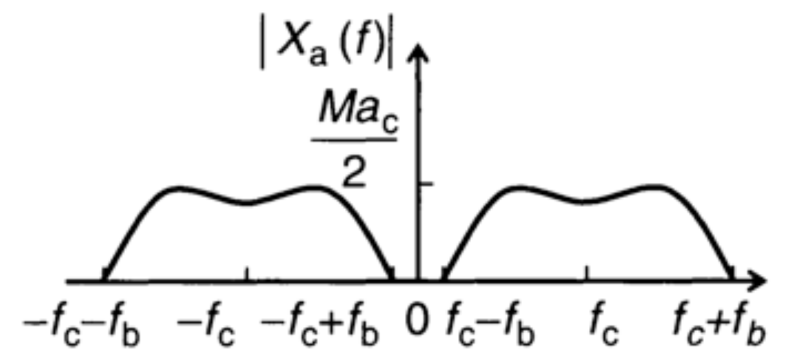
and multiply it by $2/a_c$, and then sum the left-shifted-by-f_c and right-shifted by f_c versions of the multiplied signal, the resulting signal will be





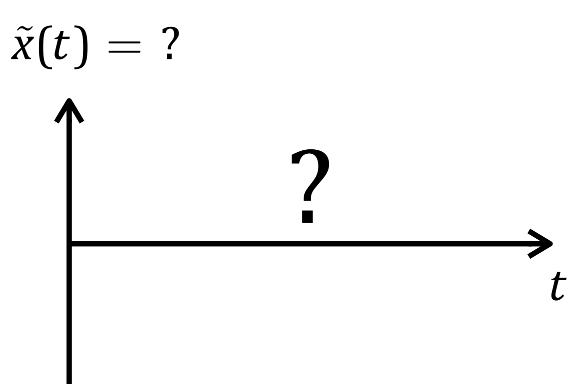
If we take

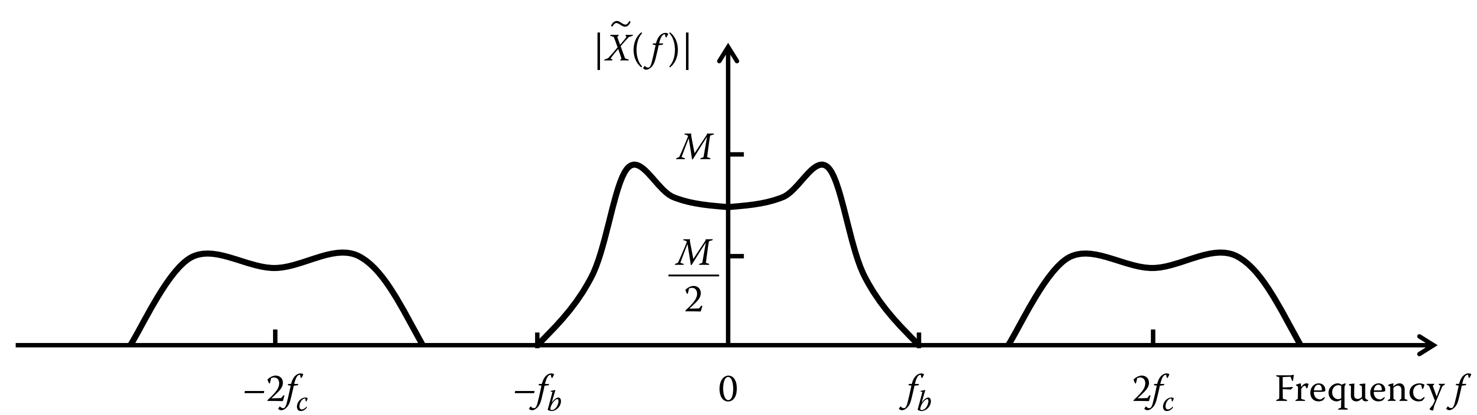




How to do this?

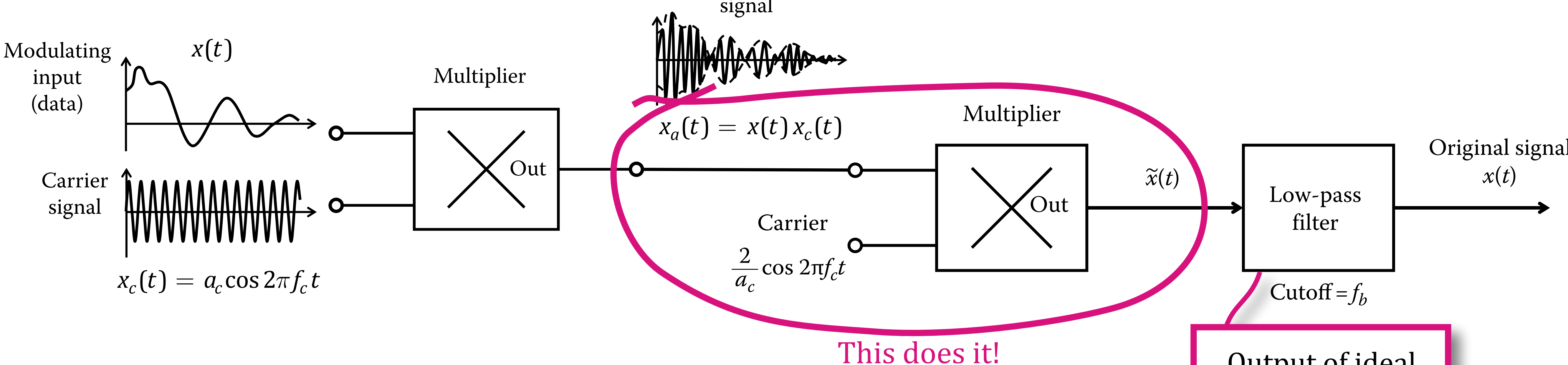
and multiply it by $2/a_c$, and then sum the left-shifted-by-f_c and right-shifted by f_c versions of the multiplied signal the resulting signal will be





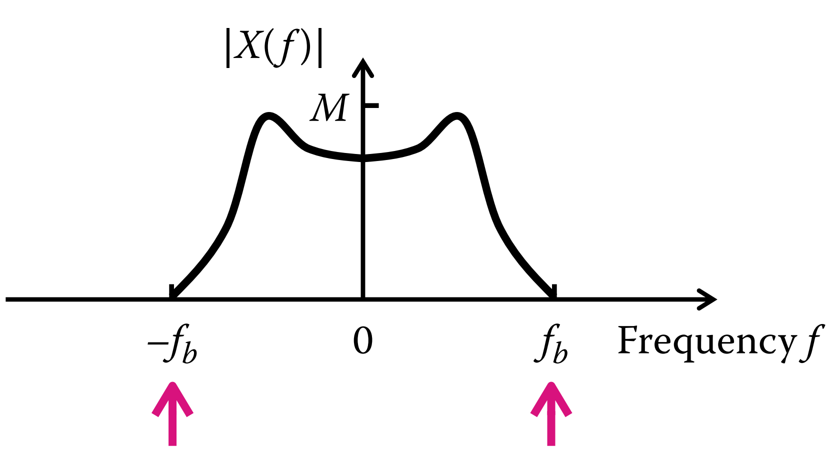
Amplitude Modulation

Demodulation

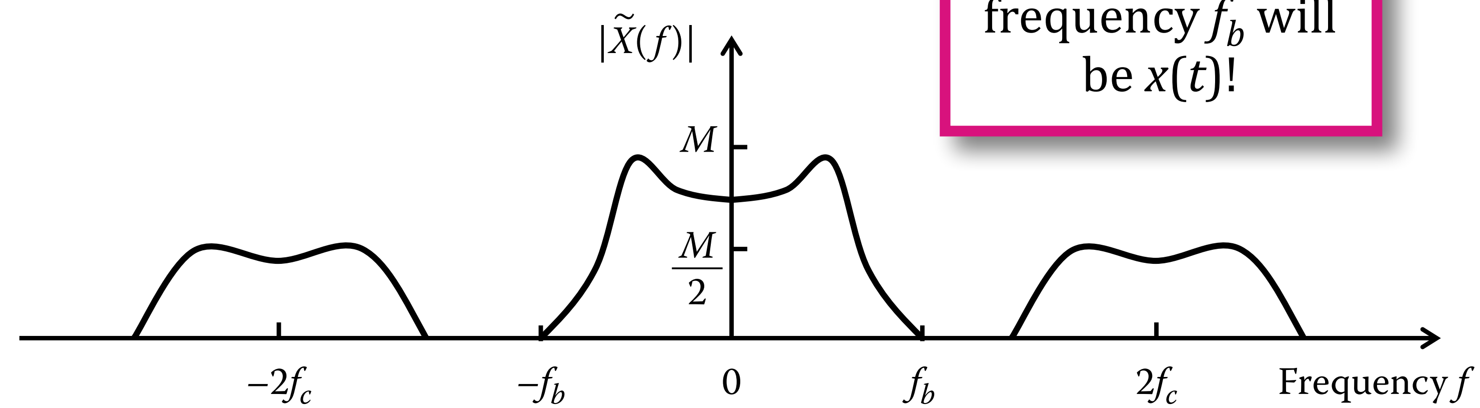


Output of ideal low-pass filter with cutoff frequency f_b will be $x(t)$!

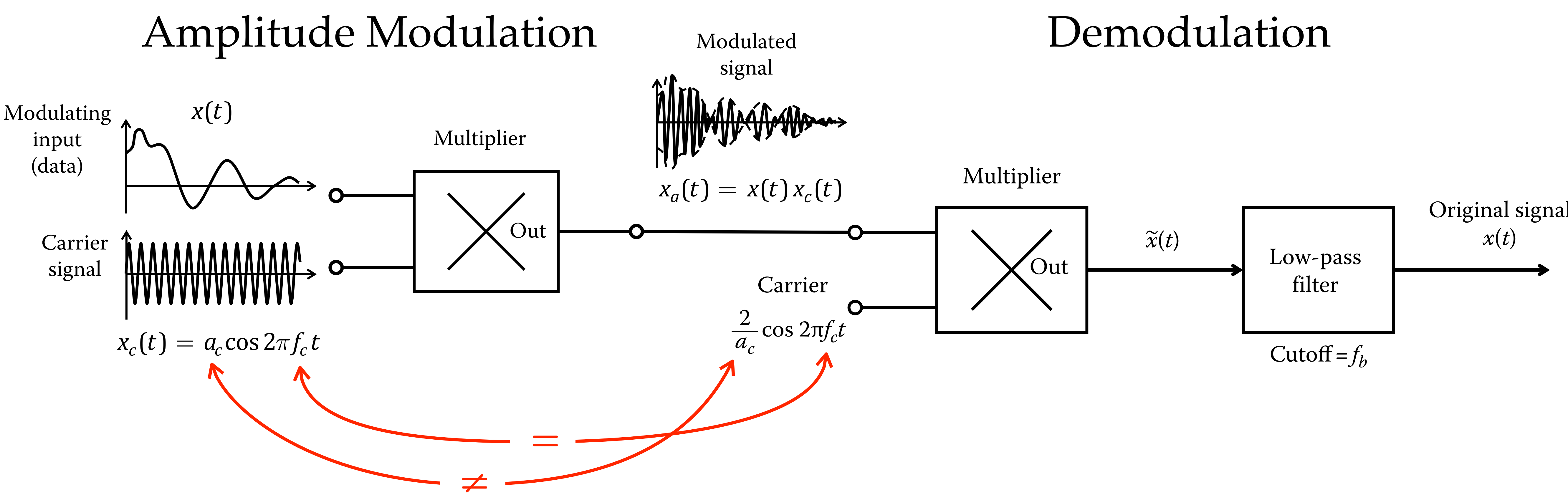
Recall that:



$\Rightarrow$



AM Radio



What conditions guarantee that the above demodulation scheme will recover $x(t)$ exactly?

f_b is the highest frequency in $x(t)$.

$$f_c > f_b$$

The low-pass filter is an ideal low pass filter.