Performance Specification and Analysis

Chapter 3

ME 473

Professor Sawyer B. Fuller

Time-Domain Performance Specifications for Response to Unit Step Command

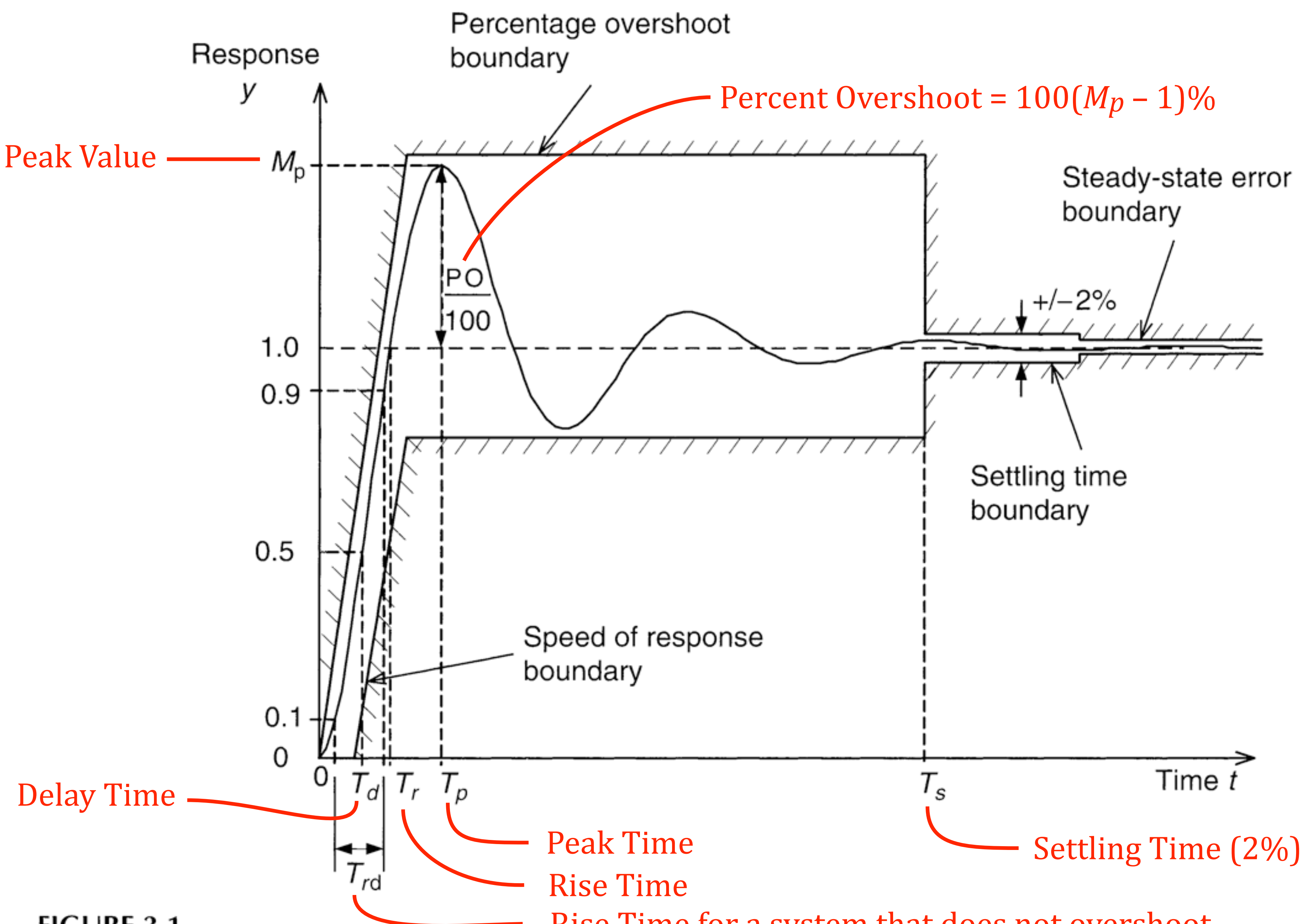


FIGURE 3.1
Response parameters for time-domain specification of performance.

TABLE 3.1

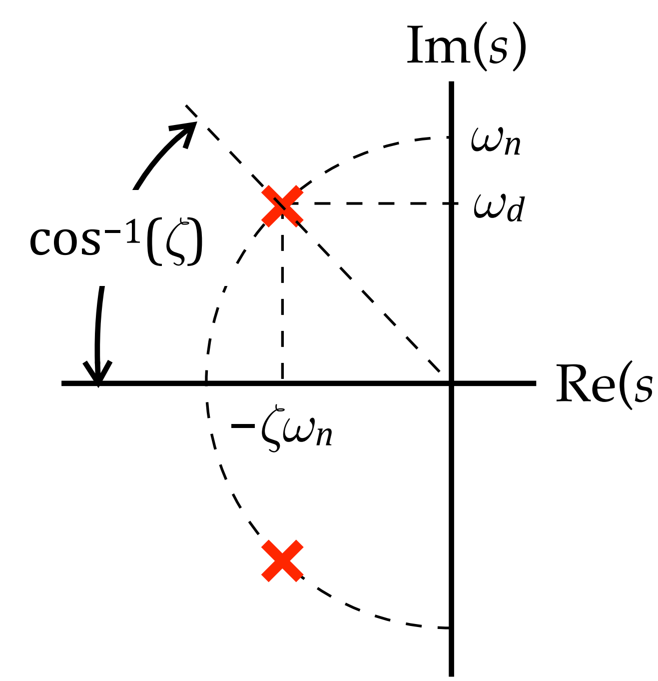
Time-Domain Performance Parameters Using the Simple Oscillator Model

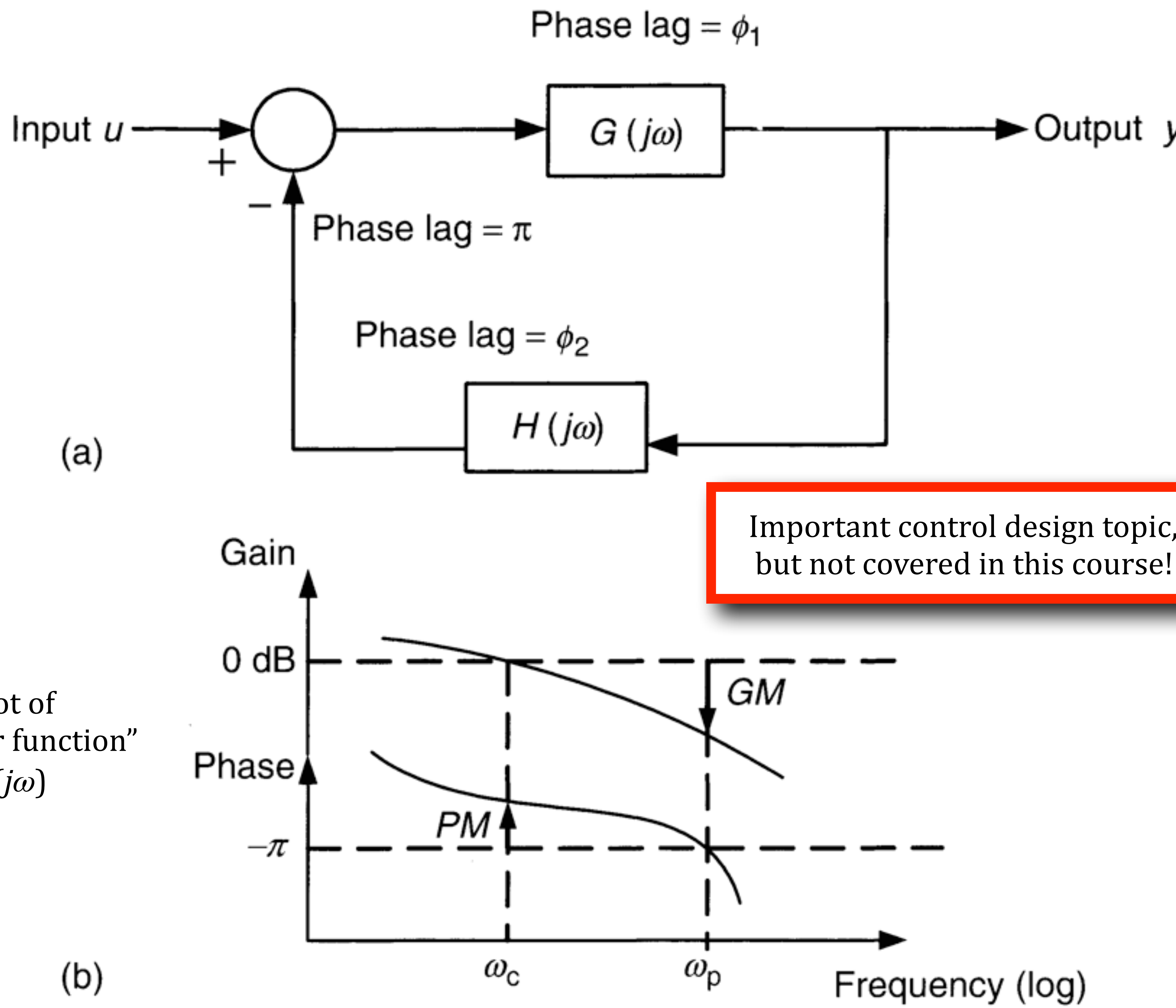
Performance Parameter	**Expression**
Rise Time	$T_{\mathrm{r}} = \dfrac{\pi - \phi}{\omega_{\mathrm{d}}}$ with $\cos\phi = \zeta$
Peak Time	$T_{\mathrm{p}} = \dfrac{\pi}{\omega_{\mathrm{d}}}$
Peak Value	$M_{\mathrm{p}} = 1 - \mathrm{e}^{-\pi\,\zeta/\sqrt{1-\zeta^2}}$
Percentage Overshoot (PO)	$\mathrm{PO} = 100\ \mathrm{e}^{-\pi\,\zeta/\sqrt{1-\zeta^2}}$
Time Constant	$\tau = \dfrac{1}{\zeta\omega_{\mathrm{n}}}$
Settling Time (2%)	$T_{\mathrm{s}} = -\dfrac{\ln\left[0.02\sqrt{1-\zeta^2}\right]}{\zeta\omega_{\mathrm{n}}} \approx 4\tau = \dfrac{4}{\zeta\omega_{\mathrm{n}}}$

Simple Oscillator Model:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u \quad \Leftrightarrow \quad u \longrightarrow \boxed{\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \longrightarrow y$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$





Important control design topic, but not covered in this course!

Bode plot of
"Loop transfer function"
$G(j\omega)H(j\omega)$

FIGURE 3.4
Illustration of gain and phase margins. (a) A feedback system. (b) Bode diagram.

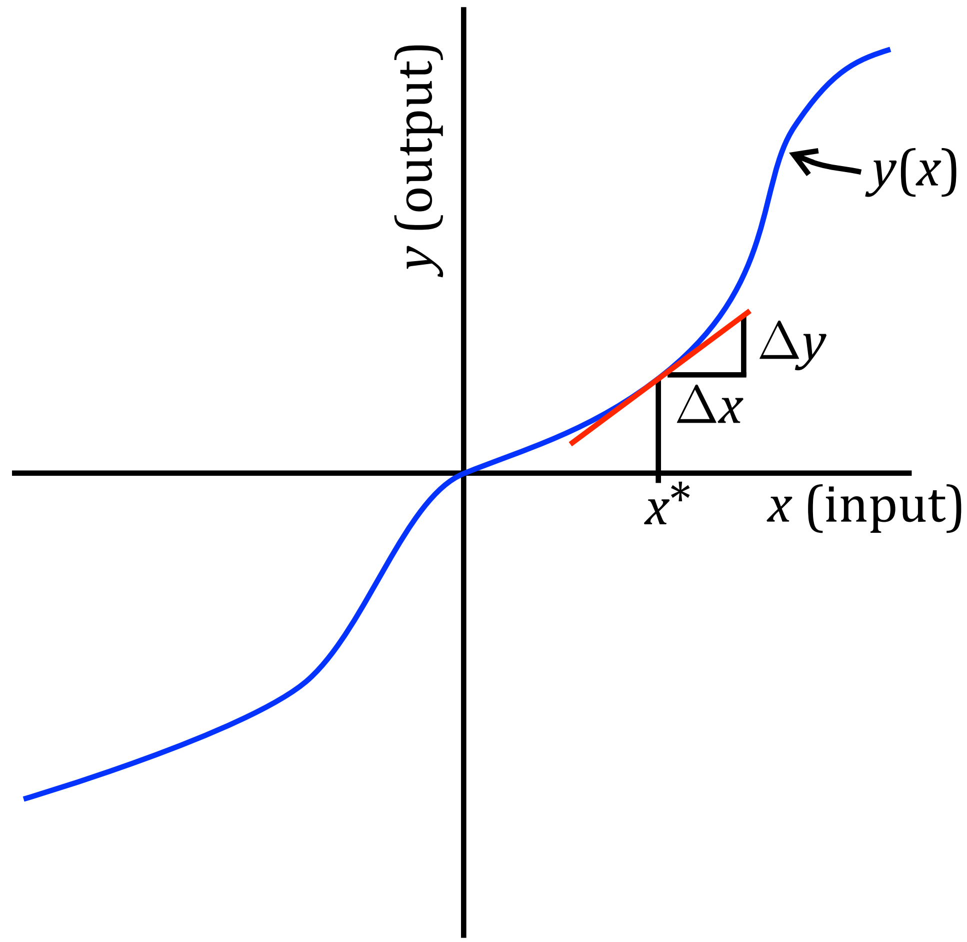
3.5.1 Rating Parameters

Typical rating parameters provided by instrument manufacturers and vendors (in their data sheets) are as follows:

- 1. Sensitivity and sensitivity error
- 2. Signal-to-noise ratio
- 3. Dynamic range
- 4. Resolution
- 5. Offset or bias
- 6. Linearity
- 7. Zero drift, full-scale drift, and calibration drift (Stability)
- 8. Useful frequency range
- 9. Bandwidth
- 10. Input and output impedances

We have already discussed the meaning and significance of some of these terms. In this section, we look at the conventional definitions given by instrument manufacturers and vendors.

Sensitivity of a device (e.g., transducer) is measured by the magnitude (peak, rms value, etc.) of the output signal corresponding to unit input (e.g., measurand). This may be expressed as the ratio of incremental output and incremental input (e.g., slope of a data curve) or, analytically, as the corresponding partial derivative. In the case of vectorial or tensorial signals (e.g., displacement, velocity, acceleration, strain, force), the direction of sensitivity should be specified.



$$\text{Sensitivity}(x) = \frac{dy}{dx}$$

$$\text{Sensitivity}(x)\Big|_{x=x^*} = \frac{\Delta y}{\Delta x}$$

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Dynamic range of an instrument is determined by the allowed lower and upper limits of its input or output (response) so as to maintain a required level of output accuracy. This range is usually expressed as a ratio (e.g., a log value in decibels). In many situations, the lower limit of dynamic range is equal to the resolution of the device. Hence, the dynamic range (ratio) is usually expressed as (range of operation)/(resolution) in dB.

Resolution of an input–output instrument is the smallest change in a signal (input) that can be detected and accurately indicated (output) by a transducer, a display unit, or any pertinent instrument. It is usually expressed as a percentage of the maximum range of the instrument or as the inverse of the dynamic range ratio. It follows that dynamic range and resolution are very closely related.

Instrument *Bandwidth* Example

(Example 3.9)

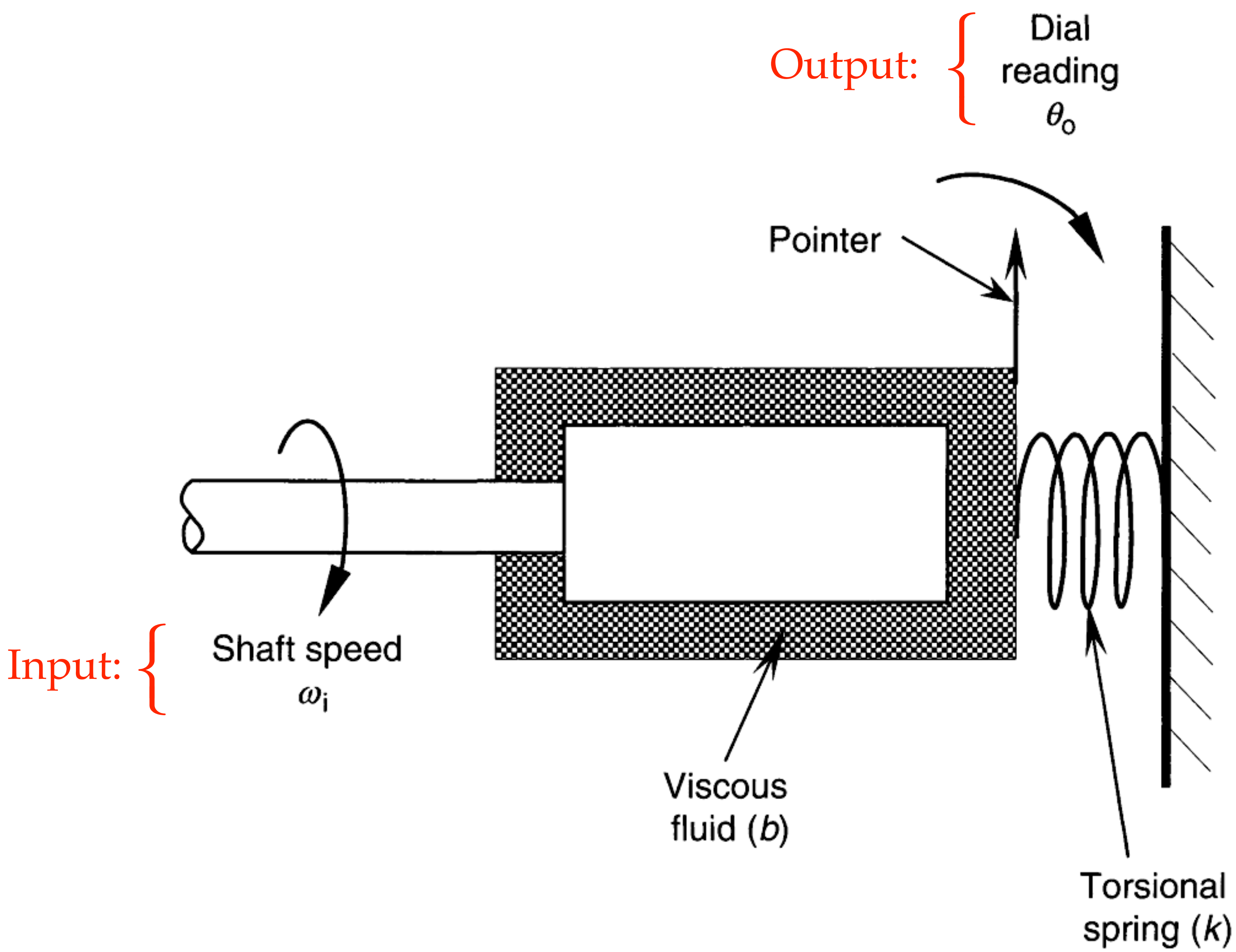
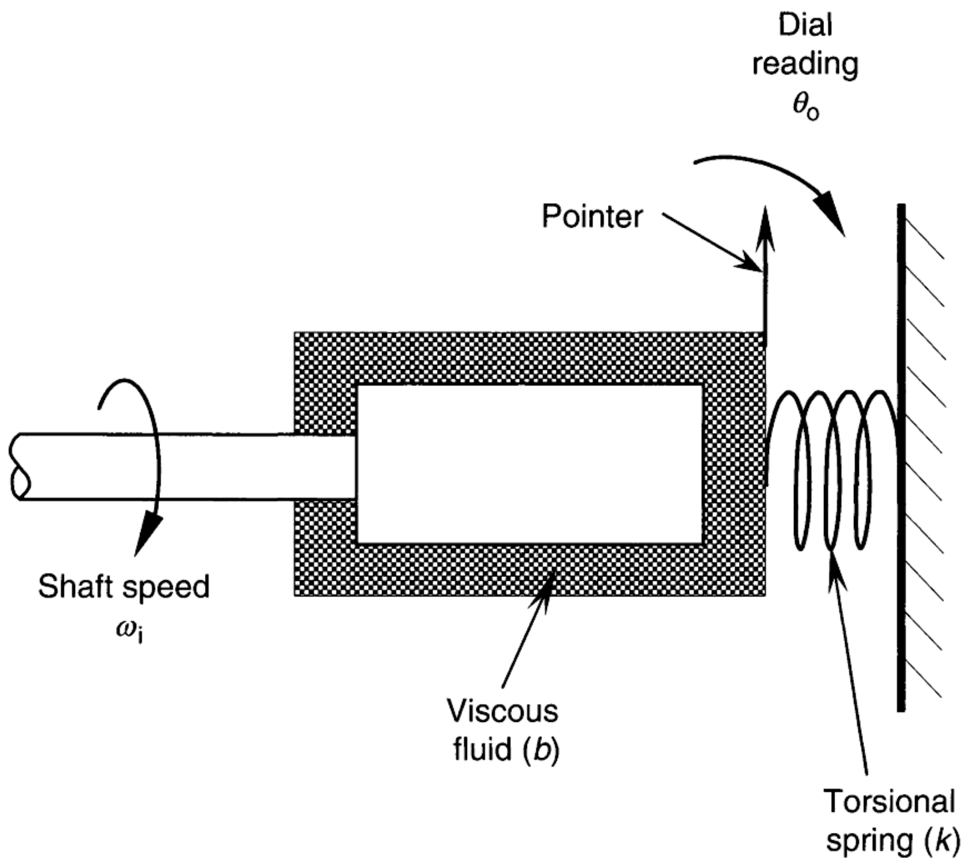


Figure 3.13
A mechanical tachometer.



Assumption: The power in via the rotating shaft goes into (1) friction losses between the rotating cylinder and the fluid, and (2) the potential energy stored in the torsional spring:

$$\underbrace{b\left(\omega_i - \dot{\theta}_o\right)}_{\text{Torque on case due to fluid friction}} = \underbrace{k\theta_o}_{\text{Torque on case due to torsional spring}}$$

Transfer function:

$$\frac{\Theta_o(s)}{\Omega_i(s)} = \frac{b}{bs+k} = \frac{b/k}{(b/k)s+1} = \frac{\tau}{\tau s+1}$$

Static Gain $= \tau$

The *half power bandwidth* is the lowest frequency at which the gain of the transfer function drops to $1/\sqrt{2}$ times its static gain.

Let

$$\omega_{bw} \triangleq \text{the half power bandwidth}$$

Then

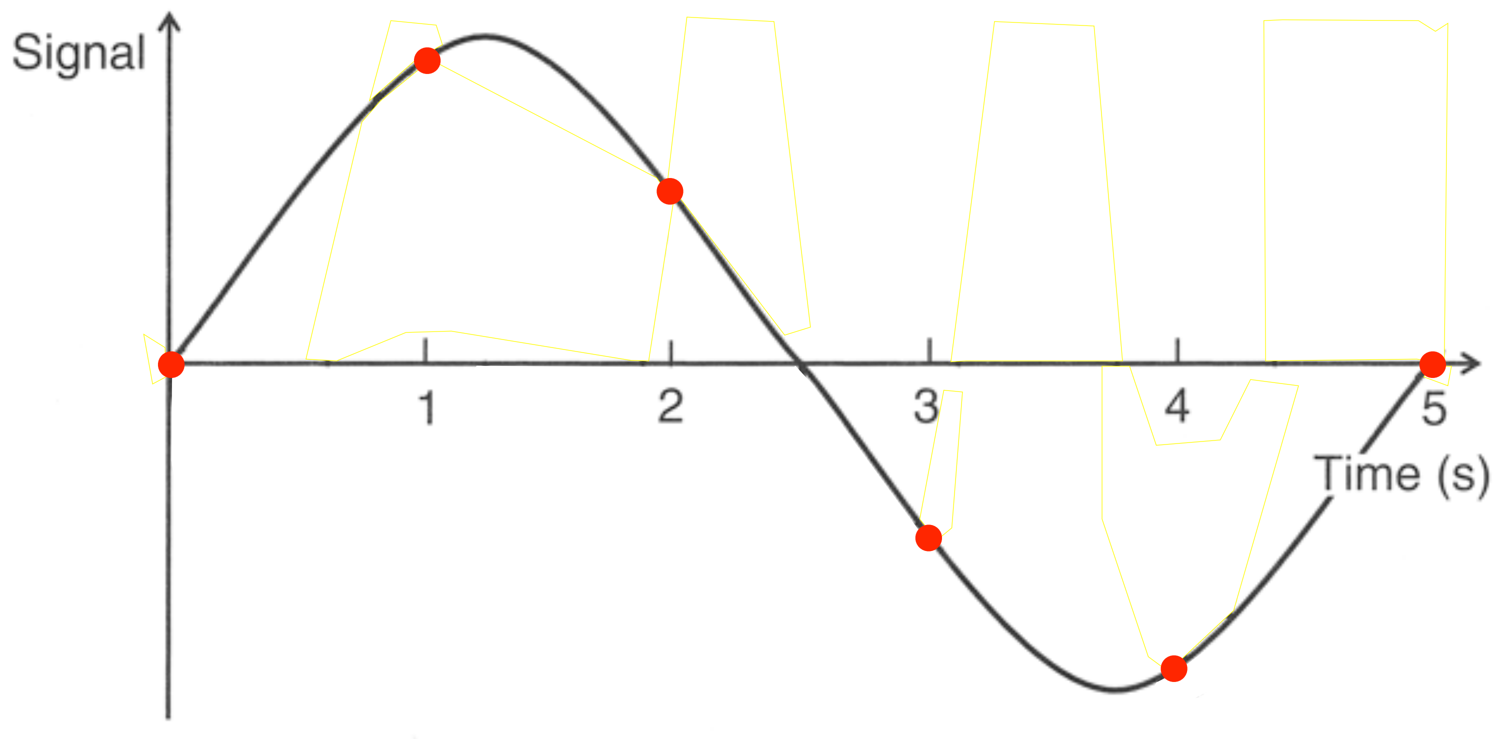
$$\frac{\text{static gain}}{\sqrt{2}} = \frac{\tau}{\sqrt{2}} = \left|\frac{\tau}{\tau(j\omega_{bw})+1}\right|$$

$$\Rightarrow \quad \frac{\tau}{\sqrt{2}} = \frac{\tau}{\sqrt{(\tau\omega_{bw})^2 + 1^2}}$$

$$\Rightarrow \quad \omega_{bw} = \frac{1}{\tau}$$

Because $20 \log\left(\frac{1}{\sqrt{2}}\right) \approx -3$ the "half power bandwidth" is also referred to as the "-3 db bandwidth".

Aliasing



Sampling rate $f_s = 1$ sample /s

Aliasing

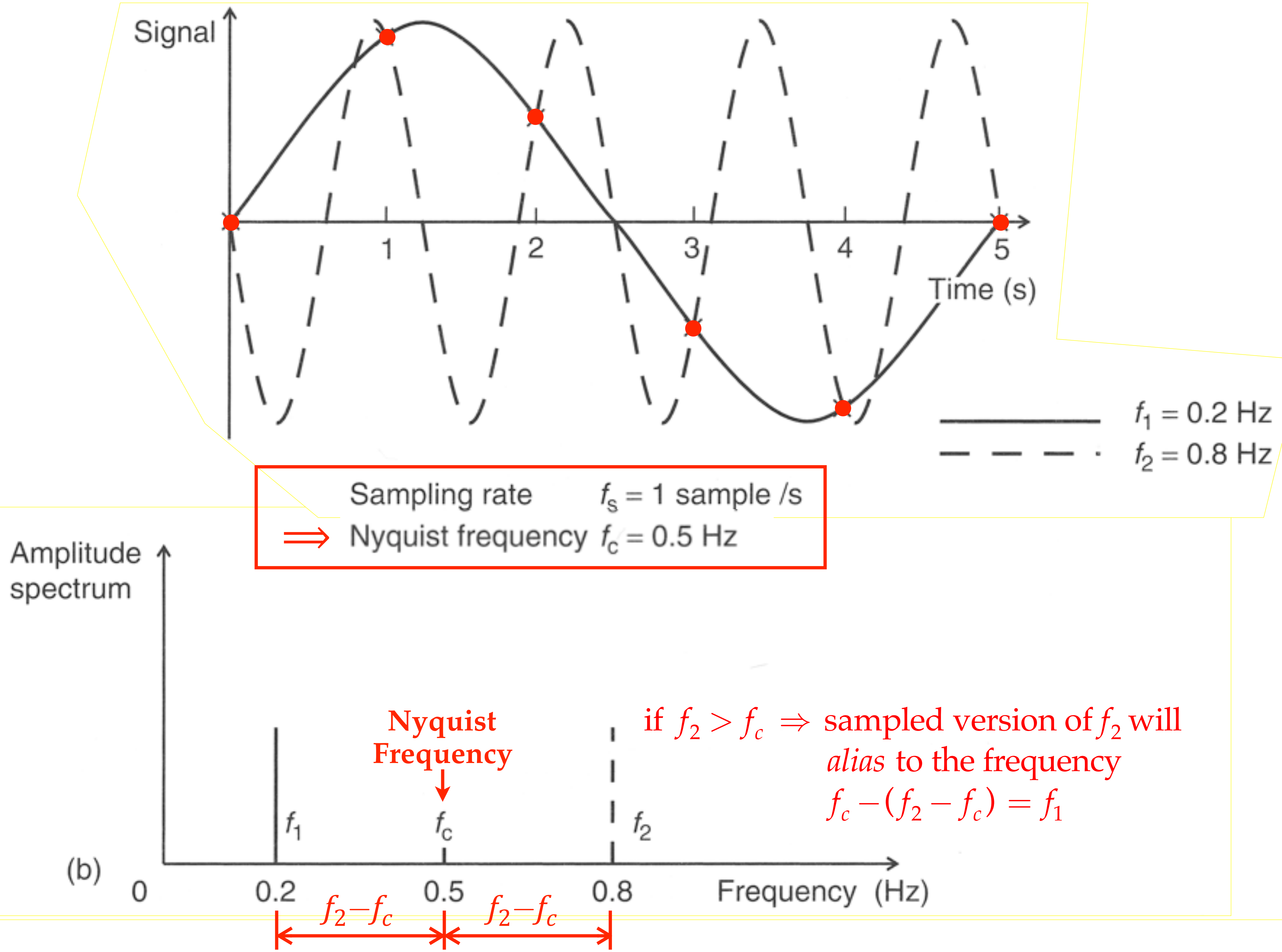
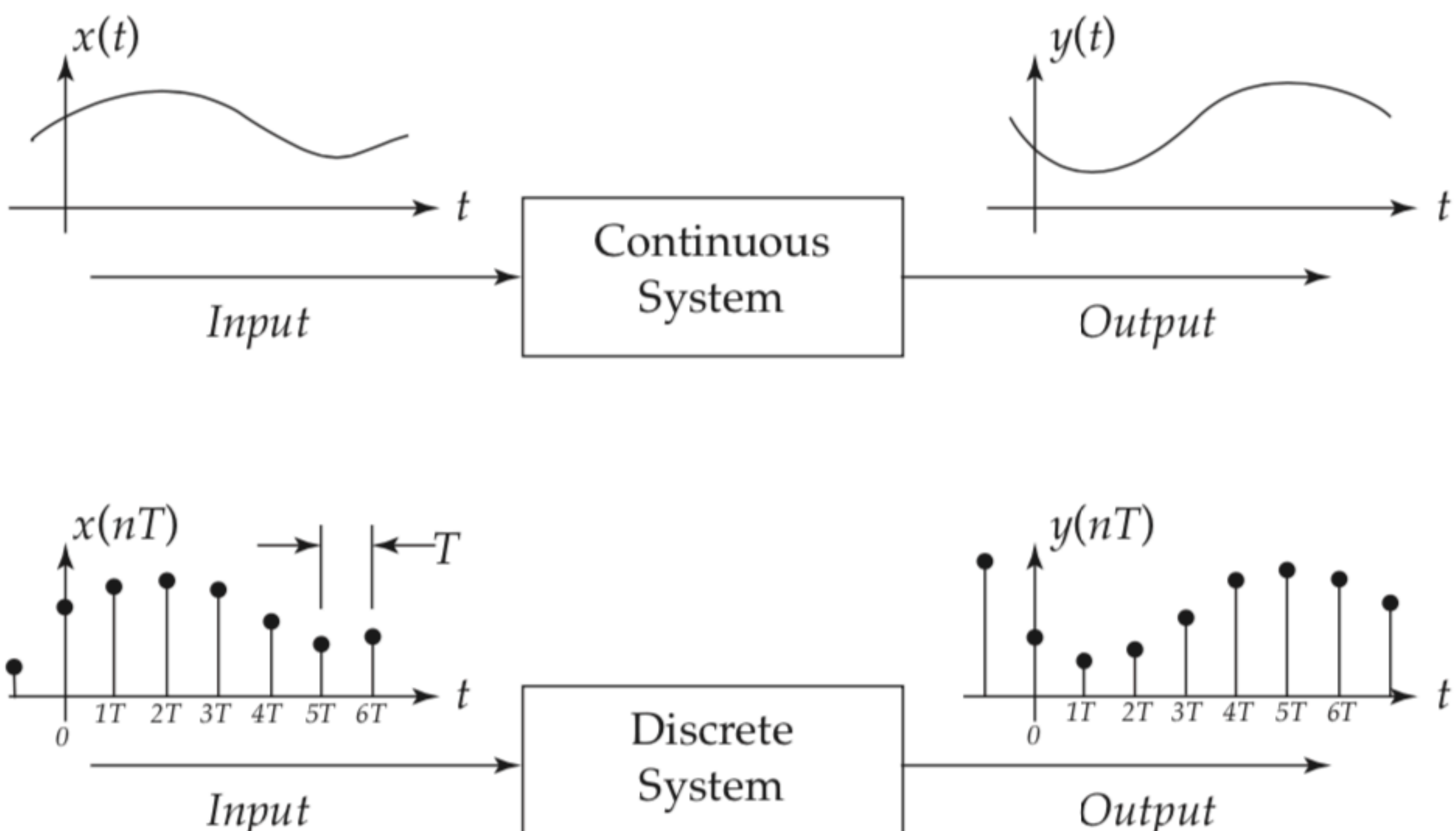


FIGURE 3.12
A simple illustration of aliasing: (a) two harmonic signals with identical sampled data. (b) frequency spectra of the two harmonic signals.

Discrete-time systems



if the system is linear, the response *y* can be described by

$$\frac{d^n y}{dt^n} + c_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + c_1\frac{dy}{dt} + c_0 y =$$
$$= d_m\frac{d^m x}{dt^m} + d_{m-1}\frac{d^{m-1}x}{dt^{m-1}} + \ldots + d_1\frac{dx}{dt} + d_0 x$$

$$\Longleftrightarrow$$

G(s)

$$a_0 y(n) + a_1 y(n-1) + \ldots + a_N y(n-N) =$$
$$= b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M)$$

$$\Longleftrightarrow$$

H(z)

- From Prof. Garbini's "Notes on discrete-time systems" (on Canvas site)

Frequency Response of Continuous-Time Systems

$$u(t) \longrightarrow \boxed{H(s)} \longrightarrow y(t)$$

If $u(t) = \sin(\omega t)$ then, in the steady state,

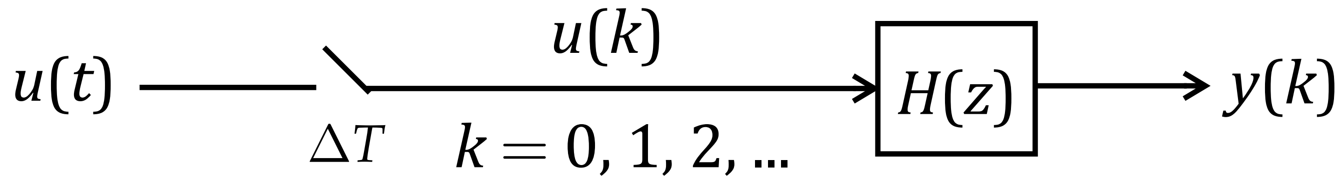
$$y(t) = \left| H(j\omega) \right| \sin\left\{ \omega t + \tan^{-1}\left(\frac{\mathrm{Im}\left[H(j\omega) \right]}{\mathrm{Re}\left[H(j\omega) \right]} \right) \right\}$$

"Frequency Response Function"

In words: The response of a linear system, with transfer function $H(s)$, in the steady-state, to the input $\sin(\omega t)$, will be a sine wave of the same frequency as the input sine wave.

The *amplitude gain* of the steady-state output sine wave relative to the input sine wave and the *phase lead angle* of the steady-state output sine wave relative to the input sine wave are easy to predict using the "frequency response function" $H(j\omega)$.

Frequency Response of Discrete-Time Systems



If $u(t) = \sin(\omega t)$, then $u(k) = \sin(\omega k \Delta T)$, for $k = 0, 1, 2, \ldots$, and, in the steady state,

$$y(k) = \left| H(e^{j\omega \Delta T}) \right| \sin\left\{ \omega\, k\, \Delta T + \tan^{-1}\left(\frac{\mathrm{Im}\left[H(e^{j\omega \Delta T}) \right]}{\mathrm{Re}\left[H(e^{j\omega \Delta T}) \right]} \right) \right\}$$

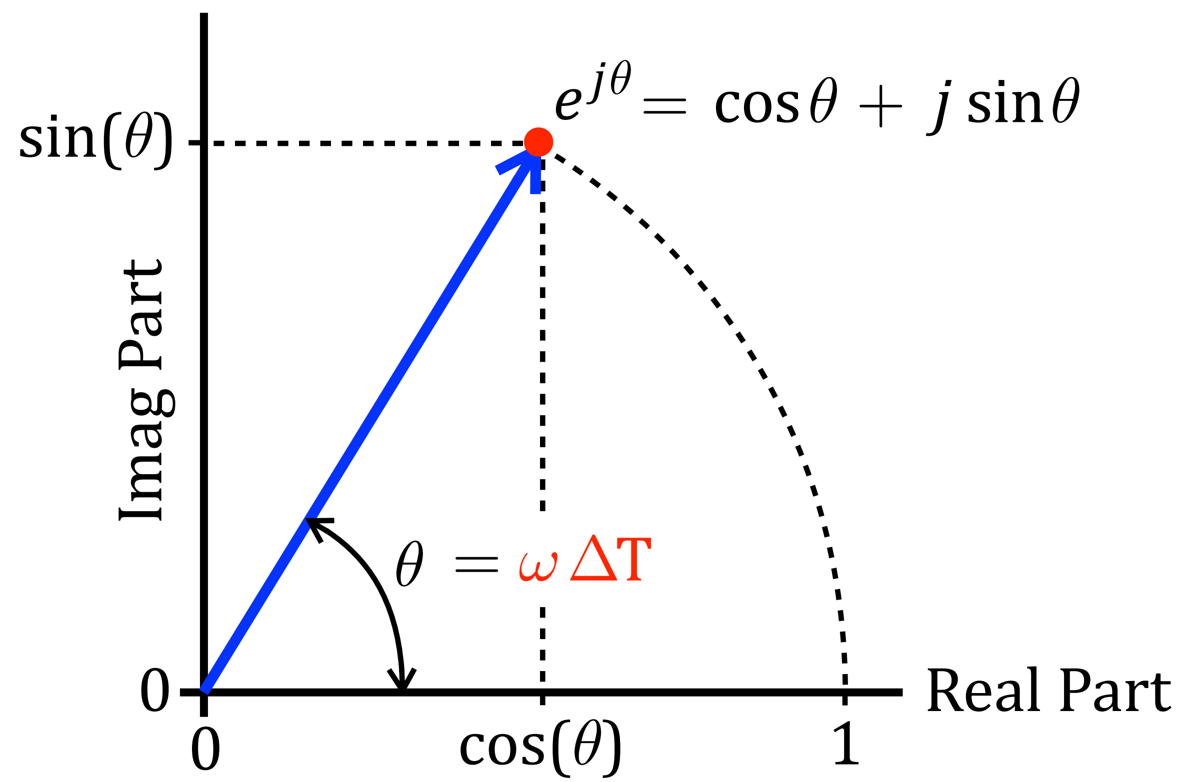
for $k = 0, 1, 2, \ldots$.

"Frequency Response Function"

The frequency response of a discrete-time system is a periodic function of the frequency ω, with period $2\pi / \Delta T$, because

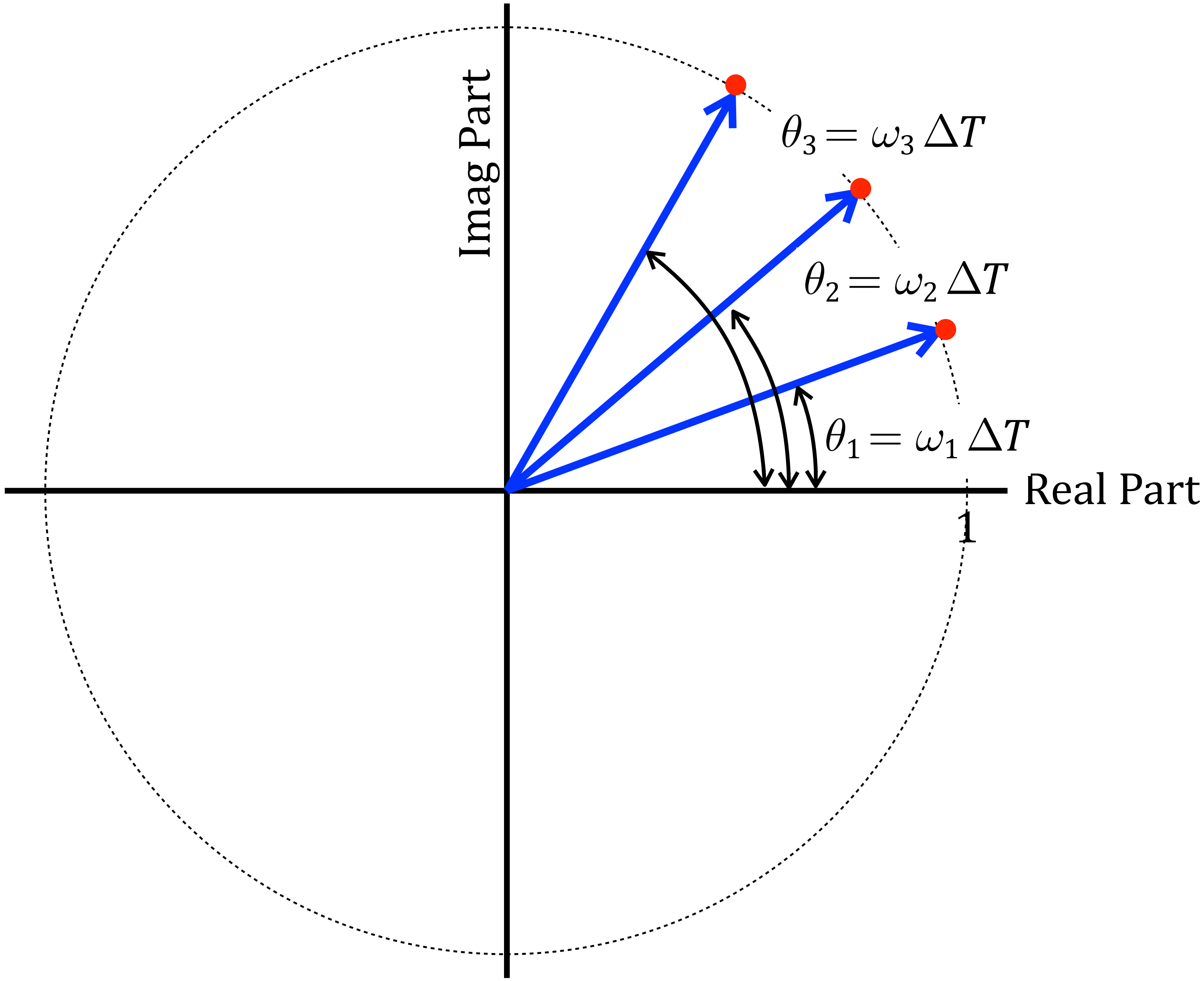
$$e^{j\omega \Delta T} = e^{j\left[\left(\omega \pm \frac{2\pi}{\Delta T}\right)\Delta T\right]} = e^{j\left[\left(\omega \pm 2\frac{2\pi}{\Delta T}\right)\Delta T\right]} = e^{j\left[\left(\omega \pm 3\frac{2\pi}{\Delta T}\right)\Delta T\right]} = \cdots$$

A graphical look at $e^{j\theta} = e^{j\omega \Delta T} \ldots$



$$y(k) = \left|H(e^{j\omega\Delta T})\right| \sin\left\{\omega k \Delta T + \tan^{-1}\left(\frac{\mathrm{Im}\left[H(e^{j\omega\Delta T})\right]}{\mathrm{Re}\left[H(e^{j\omega\Delta T})\right]}\right)\right\}$$

Generating Frequency Response Data Points



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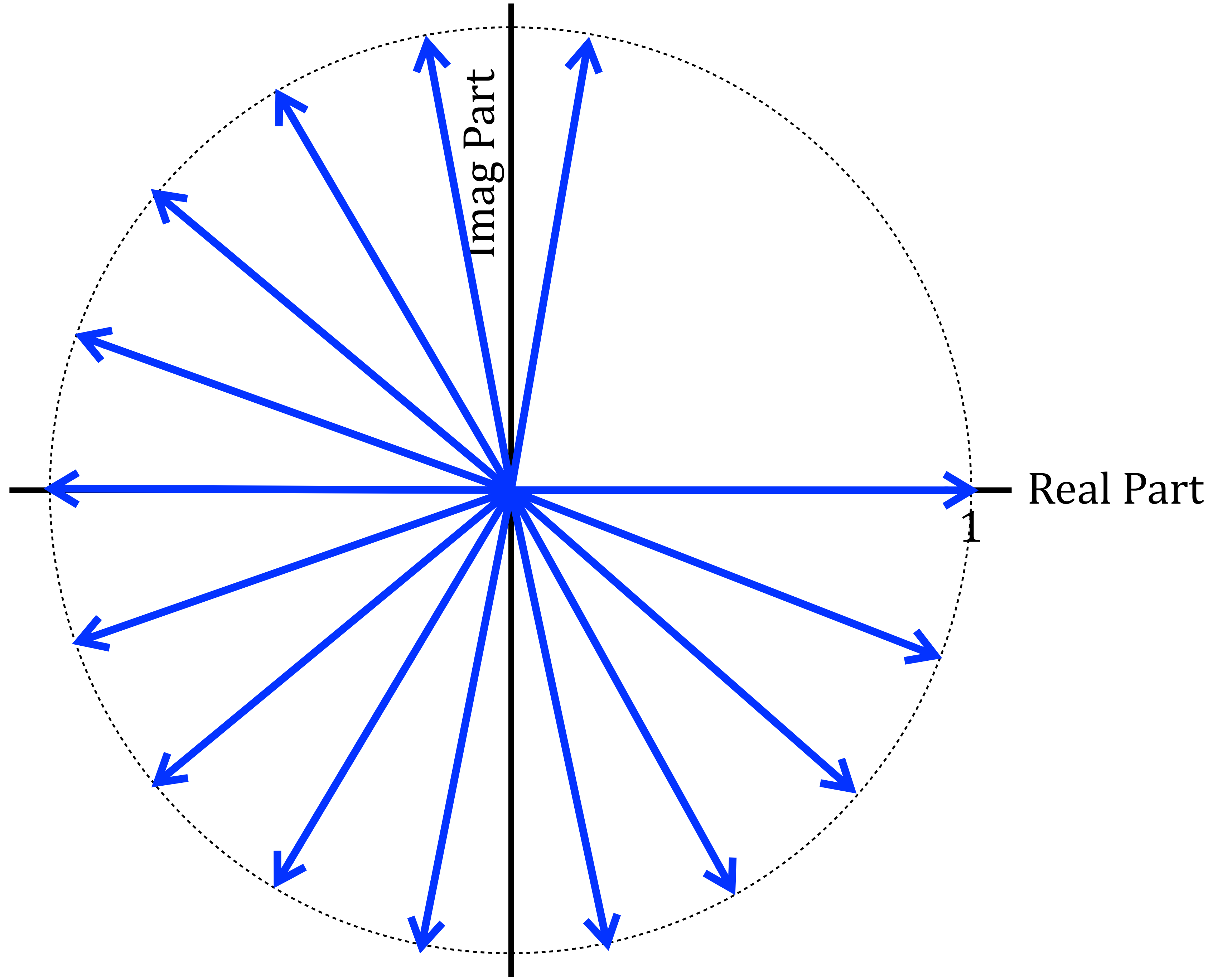

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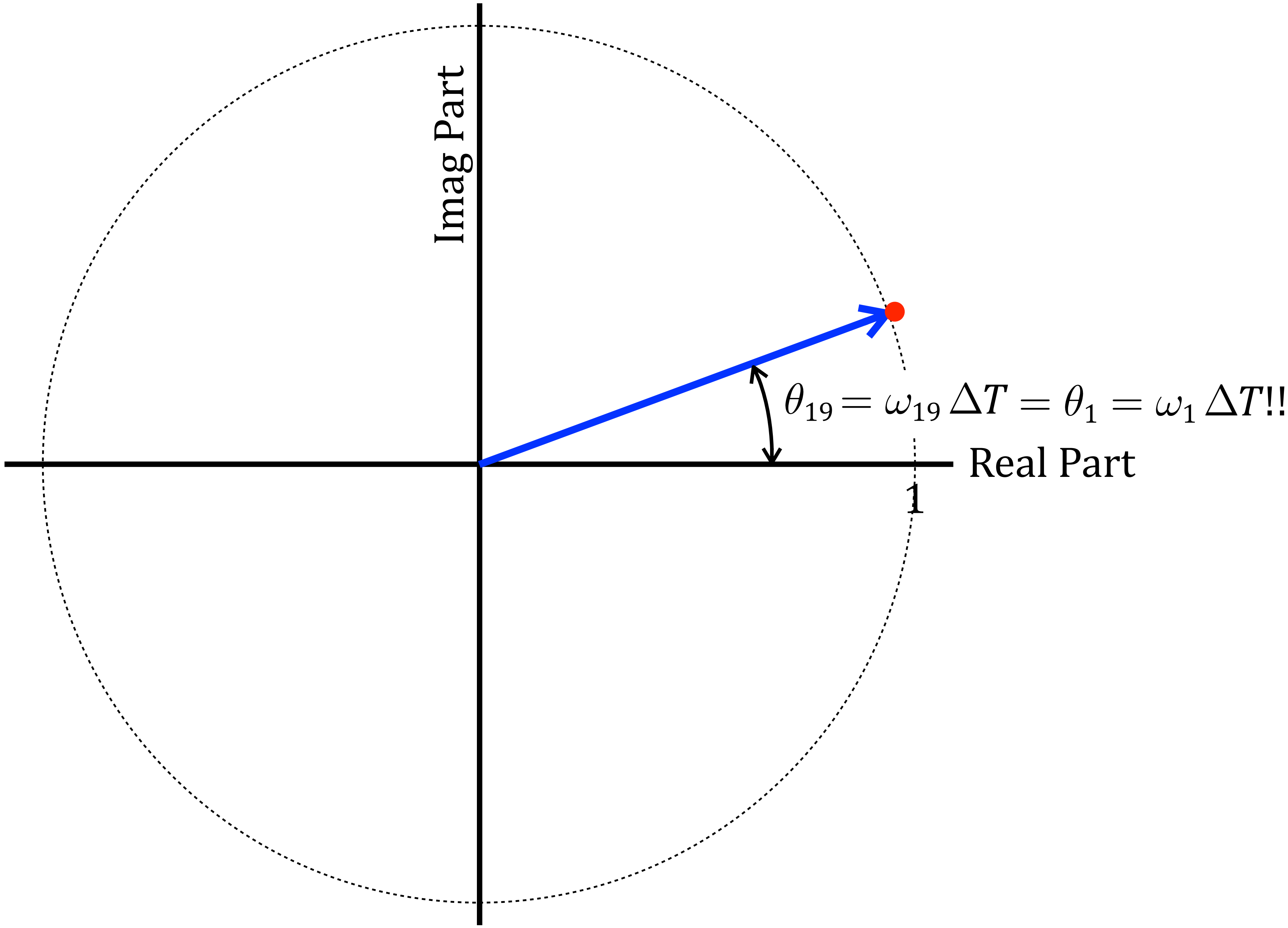

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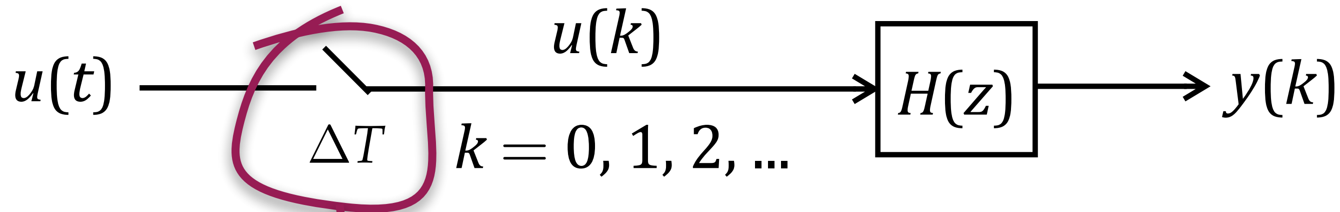
Frequency Response of Discrete-Time Systems



Responsibility for the periodicity lies here!

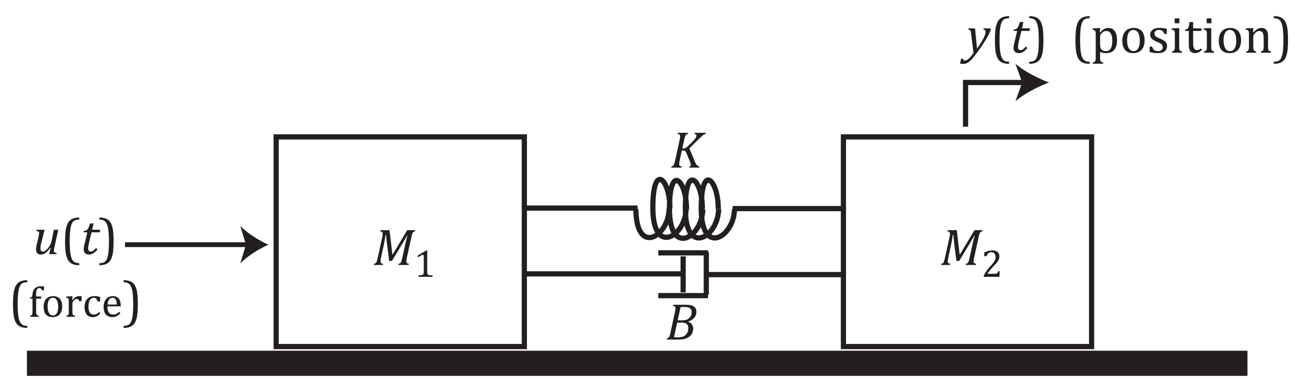
If $u(t)=\sin(\omega t)$, then $u(k)=\sin(\omega k\Delta T)$, for $k=0,1,2,\ldots$, and, in the steady state,

$$y(k)=\left|H(e^{j\omega\Delta T})\right|\sin\left\{\omega\, k\,\Delta T+\tan^{-1}\left(\frac{\mathrm{Im}\left[H(e^{j\omega\Delta T})\right]}{\mathrm{Re}\left[H(e^{j\omega\Delta T})\right]}\right)\right\}$$

for $k=0,1,2,\ldots$.

The frequency response of a discrete-time system is a periodic function of the frequency ω, with period $2\pi/\Delta T$ because

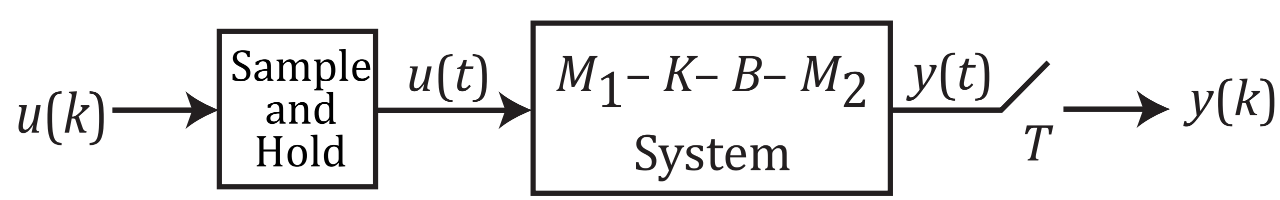
$$e^{j\omega\Delta T}=e^{j\left[\left(\omega\pm\frac{2\pi}{\Delta T}\right)\Delta T\right]}=e^{j\left[\left(\omega\pm2\frac{2\pi}{\Delta T}\right)\Delta T\right]}=e^{j\left[\left(\omega\pm3\frac{2\pi}{\Delta T}\right)\Delta T\right]}=\cdots$$



No friction at sliding surfaces

$M_1 = 1$ kg $M_2 = 1$ kg $K = 200\,\pi^2$ N/m $B = 0.2\,\pi$ N/m/sec

Filter for Continuous-Time Signals



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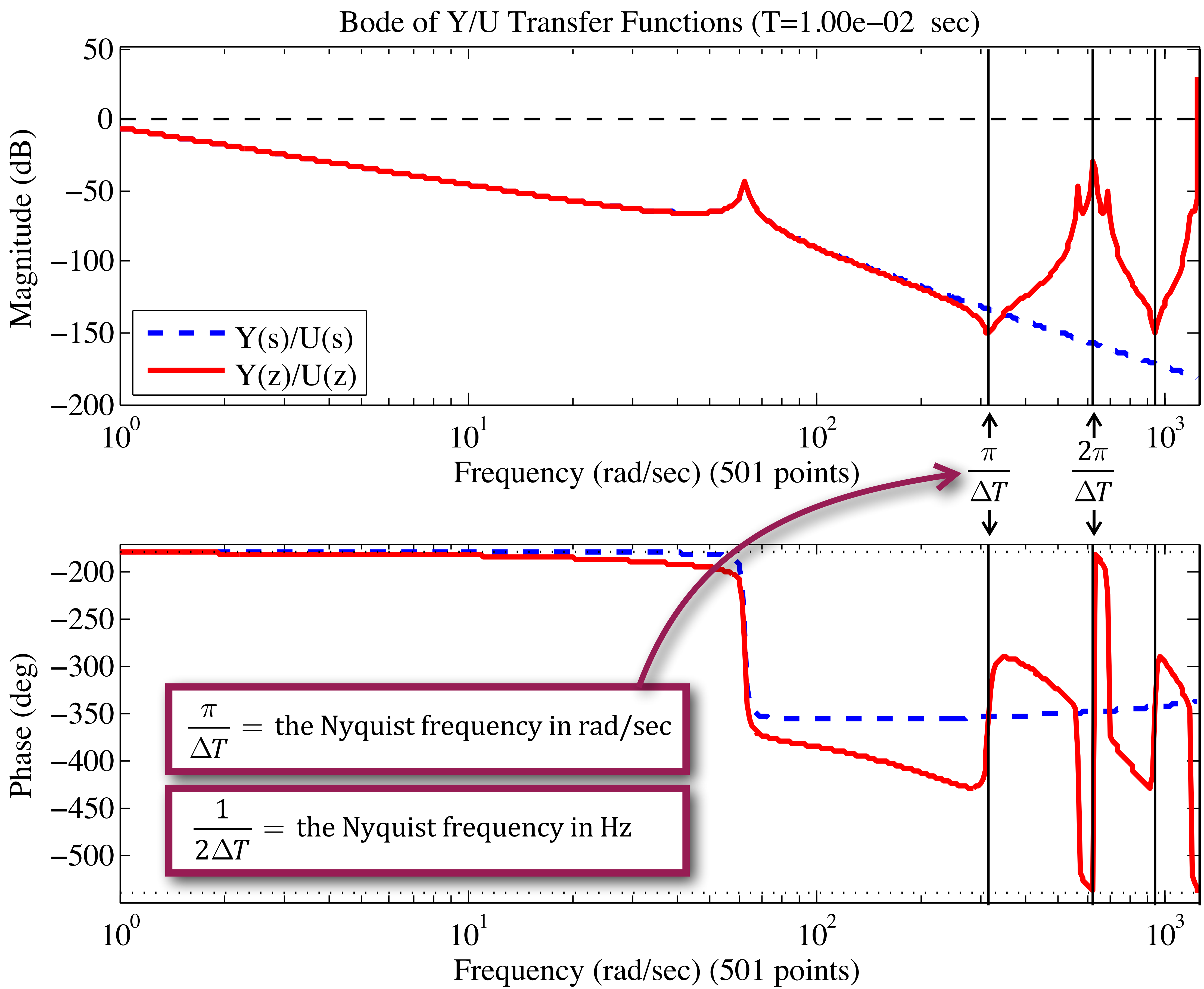
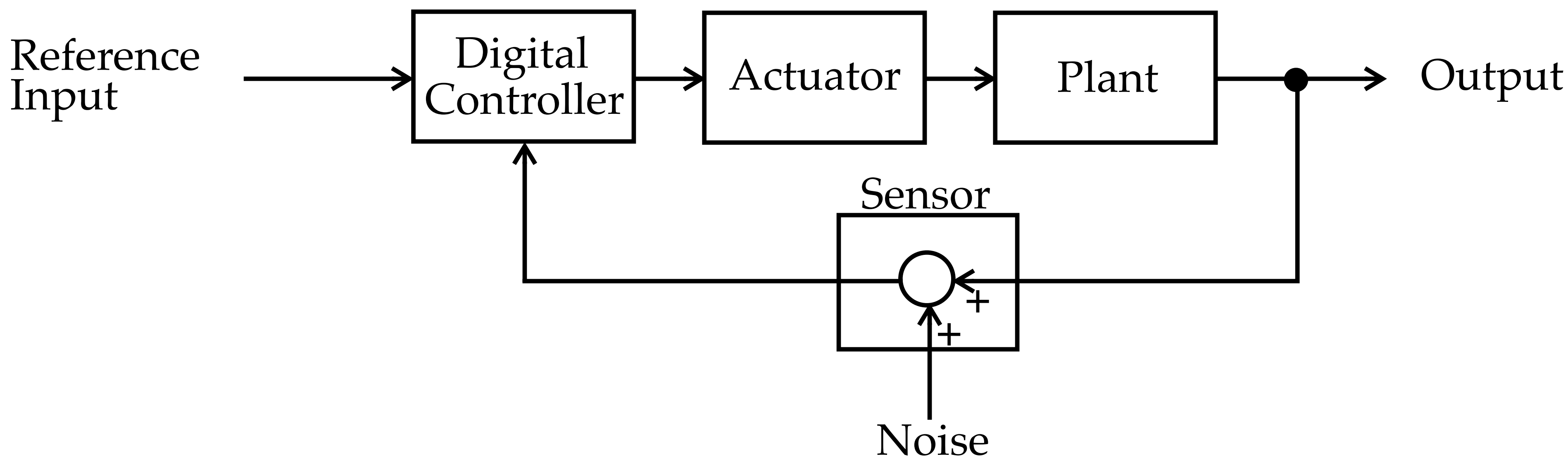

Bode of Y/U Transfer Functions (T=1.00e−02 sec)
Magnitude (dB)
Y(s)/U(s)
Y(z)/U(z)
Frequency (rad/sec) (501 points)
$\frac{\pi}{\Delta T}$
$\frac{2\pi}{\Delta T}$
Phase (deg)
$\frac{\pi}{\Delta T}$ = the Nyquist frequency in rad/sec
$\frac{1}{2\Delta T}$ = the Nyquist frequency in Hz
Frequency (rad/sec) (501 points)

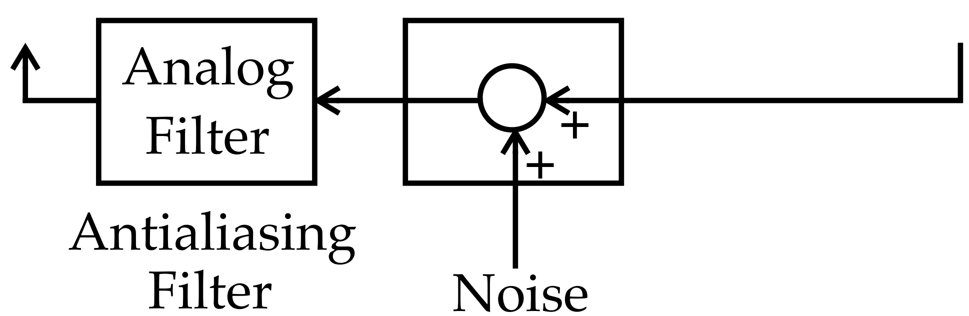
Closed-Loop Control System



Can Digital Controller filter out the sensor noise?

Practically impossible if sensor noise frequencies above Nyquist frequency!

solution: add an "antialiasing filter" to remove high-frequency noise:



3.7.1 Sampling Theorem

If a time signal $x(t)$ is sampled at equal steps of ΔT, no information regarding its frequency spectrum $X(f)$ is obtained for frequencies higher than $f_c = 1/(2\Delta T)$. This fact is known as Shannon's sampling theorem, and the limiting (cutoff) frequency is called the *Nyquist frequency*.

It can be shown that the aliasing error is caused by folding of the high-frequency segment of the frequency spectrum beyond the Nyquist frequency into the low-frequency segment.

$$u(t) \xrightarrow[\Delta T]{} y(k)$$

$$k = 0, 1, 2, \ldots$$

$$\frac{1}{2\Delta T} = \text{The Nyquist Frequency in Hertz} \qquad \frac{\pi}{\Delta T} = \text{The Nyquist Frequency in rad/sec}$$

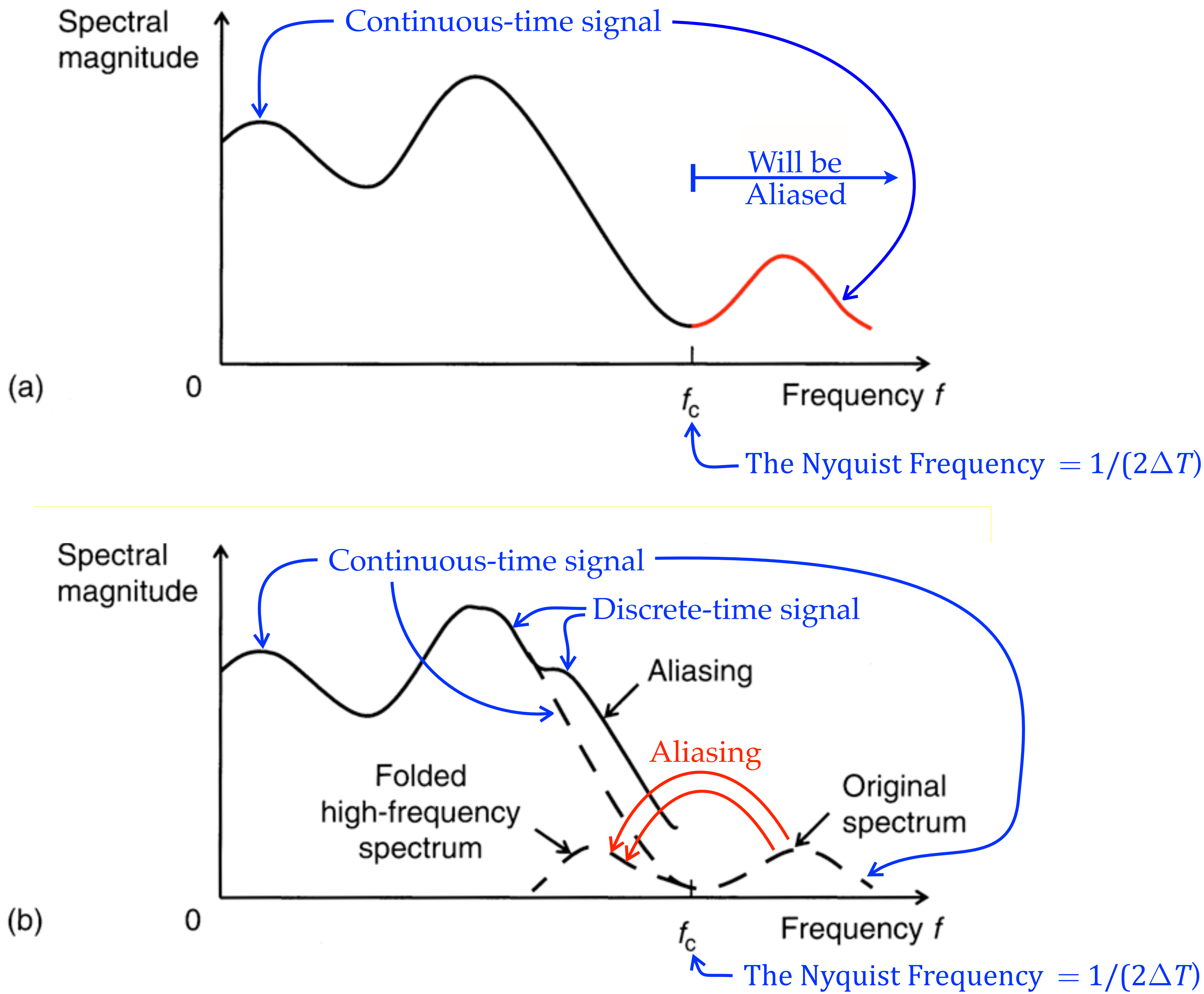
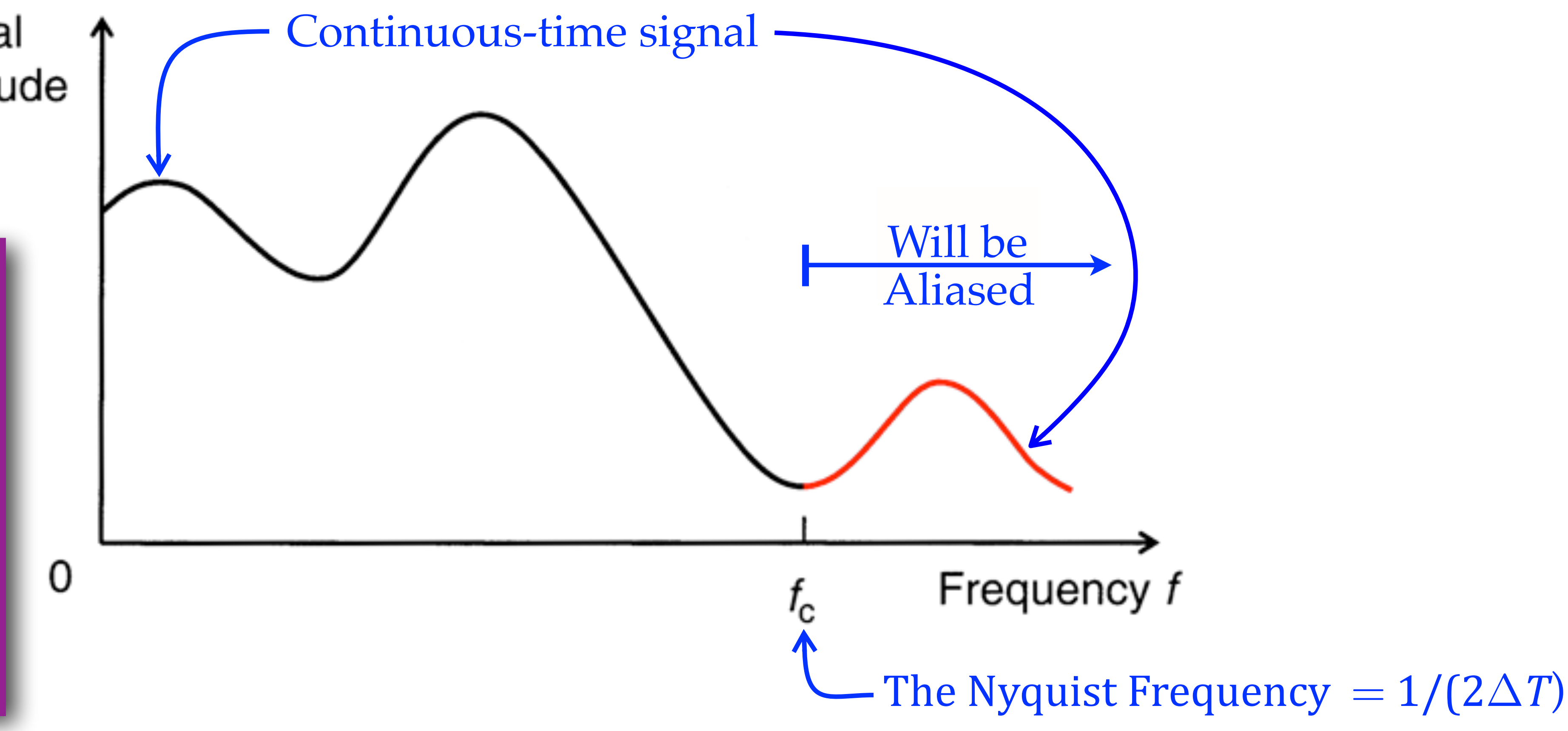
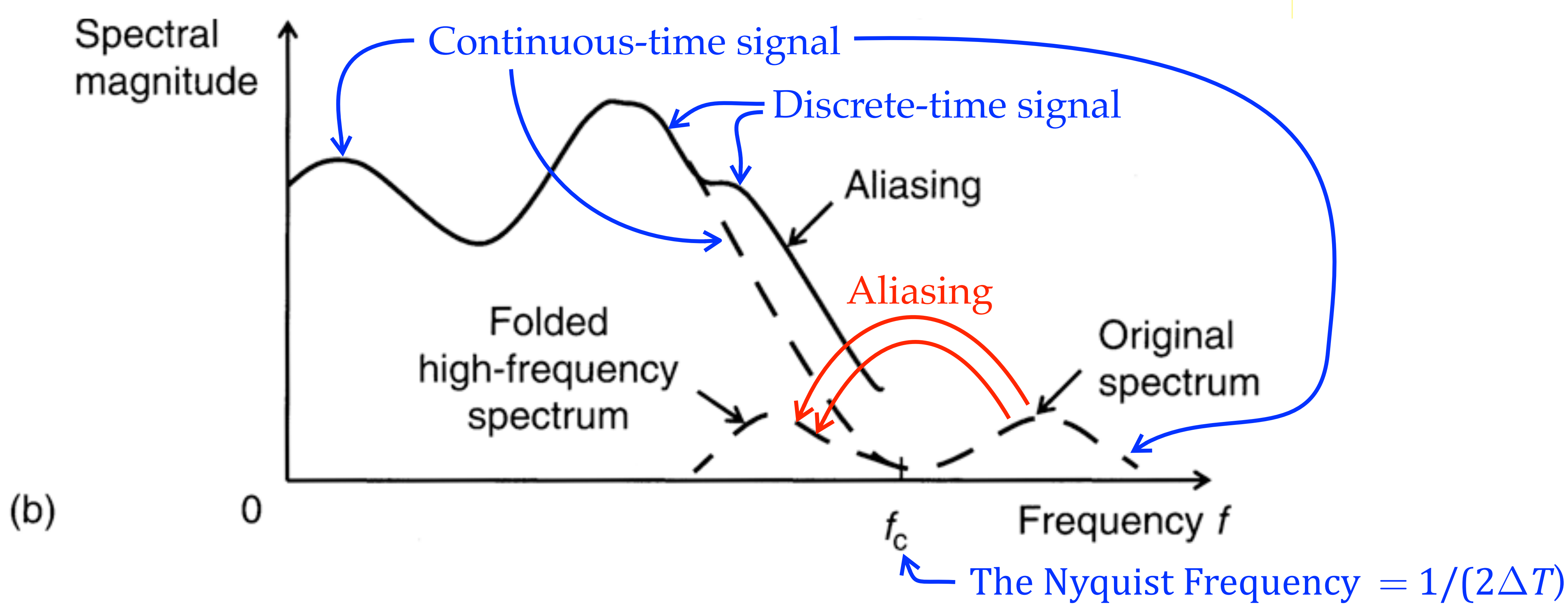


Figure 3.14
Aliasing distortion of a frequency spectrum. (a) Original spectrum. (b) Distorted spectrum due to aliasing.

Conversely, if the continuous-time signal has no frequency content above the Nyquist frequency, then it can be shown that it can be recovered *exactly* from its sampled values.





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