Introduction to differential equations and applications

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Prolegomenon

These are the lecture notes for Amath 351: Introduction to differential equations and applications. This is the first year these notes are typed up, thus it is guaranteed that these notes are full of mistakes of all kinds, both innocent and unforgivable. Please point out these mistakes to me so they may be corrected for the benefit of your successors. If you think that a different phrasing of something would result in better understanding, please let me know.

The figures in these lectures were produced using John Polking's DFIELD2005.10 and PPLANE2005.10 (see `http://math.rice.edu/~dfield/dfpp.html`), as well as Maple (see `http://www.maplesoft.com`) and lots of xfig.

Contents

Lecture 1. Differential equations and their solutions

1. Algebraic equations

An algebraic equation is an equation between an unknown quantity x and functions of this quantity x. It may be written in the form

$$F(x) = 0,$$

such as

$$2x^2 + x - 3 = 0.$$

If there are multiple variables, say x and y, then the equation is of the general form

$$F(x, y) = 0,$$

where F is a vector. As an example,

$$\left\{ \begin{array}{rcl} x^2 - y^2 & = & 2, \\ x + y & = & \cos(x) \end{array} \right. .$$

So, given an algebraic equation, or a set of algebraic equations, we need to find a number or a set of numbers.

2. Solutions of algebraic equations

Claim: $x = 4$ is a solution of $x^2 - 2x - 8 = 0$.
Check: Plug $x = 4$ into the equation: $4^2 - 2 * 4 - 8 = 0$, which is true. Thus indeed, $x = 4$ solves the equation.

Note that checking that a given number solves the algebraic equation requires no more effort than plugging the proposed solution into the equation. This is a lot easier than actually finding a solution.

3. Differential equations

A differential equation is a relationship between a function and its derivatives. It asks us to find a function, instead of a number.

Example: $y' + 2y = 1$, where $y = y(x)$.
In order to solve this equation, we need to find **all** functions that satisfy it.
Claim: $y = \frac{1}{2} + ce^{-2x}$ is a solution of this equation, with c being a constant.
Check: PLUG IT IN!

$$
\begin{aligned}
& & y' &= -2ce^{-2x}, \\
\Rightarrow & & -2ce^{-2x} + 2\left(\frac{1}{2} + ce^{-2x}\right) &\stackrel{?}{=} 1 \\
\Rightarrow & & 1 &\stackrel{!}{=} 1.
\end{aligned}
$$

We should not be surprised there is an arbitrary constant in the solution. The differential equation contains one derivative. To get rid of it, you will need to integrate at some point. This integration will result in an integration constant.

This is pretty good: although we don't know yet how to solve differential equations, we already know how to verify that something is a solution. Note that this verification requires us to take derivatives. That's okay: taking derivatives is mechanical. There's a set of rules, and if we follow these rules, we're doing fine. Since solving differential equations requires us to get rid of derivatives, you might justifiably think that integration enters into it. But integration is a lot harder than differentiation: there are some rules, but often there are tricks to be used. Even more often, integrals cannot be explicitly done. So be it. In summary:

$$\boxed{\textbf{Checking = Plugging in!}}$$

4. The order of a differential equation

The order of a differential equation is the order of the highest derivative appearing in it.

Example: $y' + 2y = 1$ is a first-order equation.

Example: $y''' = y'' - y + \sin(x)$ is a third-order equation.

Ordinary vs. partial differential equations

If a differential equation contains derivatives with respect to only one variable, it is called an ordinary differential equation. Otherwise it is called a partial differential equation.

Example: $y' + 2y = 1$ is an ordinary differential equation.

Example: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ is a partial differential equation.

5. Linear vs. nonlinear differential equations

An equation is called **linear** if the unknowns in it appear in a linear way: they do not multiply each other or themselves, and they do not appear as arguments of nonlinear functions.

Example: $y' + 2y = 1$ is a linear differential equation

Example: $y' = y^2$ is a nonlinear differential equation, because of the y^2 term.

Example: $y' = \frac{1}{1+y}$ is a nonlinear differential equation, because the y on the right-hand side appears in the nonlinear function $1/(1 + y)$.

Example: $yy' = x+y$ is a nonlinear differential equation, because the unknown functions y and y' multiply each other in the left-hand side.

Example: check that $y = c_1 \cos(x) + c_2 \sin(x)$ is a solution of $y'' + y = 0$. Note that this is a linear second-order equation.

Check: Plugging in gives:

$$\begin{aligned} y' &= -c_1 \sin(x) + c_2 \cos(x) \\ y'' &= -c_1 \cos(x) - c_2 \sin(x) \\ \Rightarrow \quad -c_1 \cos(x) - c_2 \sin(x) + (c_1 \cos(x) + c_2 \sin(x)) &\stackrel{?}{=} 0 \\ \Rightarrow \quad 0 &\stackrel{!}{=} 0. \end{aligned}$$

Note that the solution of this second-order problem depends on two arbitrary constants c_1 and c_2. This is to be expected: in order to get rid of two derivatives, you expect to have to integrate twice, resulting in two integration constants.

6. Initial-value problems

An **initial-value problem** is a differential equation together with some algebraic conditions which allow you to determine the arbitrary constants. In general, if you want to determine all arbitrary constants, you need to specify as many **initial conditions** as the order of the equation.

Example: show that $y = \frac{1}{2} + \frac{1}{2}e^{-2x}$ satisfies the initial-value problem

$$\begin{cases} y' + 2y &= 1 \\ y(0) = 1 & \end{cases}.$$

We have already verified that the given y satisfies the differential equation, so it is left to check that it satisfies the initial condition. At $x = 0$, $y = 1/2 + 1/2 = 1$, so the initial condition is indeed satisfied.

Example: find c_1, c_2 so that $y = c_1 \cos(x) + c_2 \sin(x)$ satisfies the initial-value problem

$$\begin{cases} y'' + y &= 0 \\ y(0) = 1, y'(0) &= 0 \end{cases}.$$

We already know that the given $y(x)$ satisfies the differential equation for all choices of c_1 and c_2. We have $y(0) = c_1$, so that from the first initial condition it follows that $c_1 = 1$. Next, $y' = -c_1 \sin(x) + c_2 \cos(x)$, so that $y'(0) = c_2$. It follows from the second initial condition that $c_2 = 0$, so that the solution to the initial-value problem is $y = \cos(x)$.

Geometry, the previous example may be understood in terms of curves and tangents. This is illustrated in Fig. 1. The first initial condition specifies where the curve starts at $x = 0$, whereas the second one specifies how it starts there: in this case horizontally, since the derivative is zero.

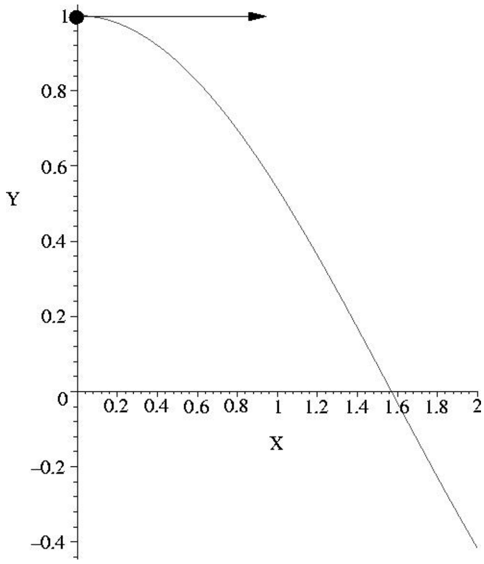

Figure 1: The solution of the initial-value problem $y'' + y = 0$, with $y(0) = 1$ (the black circle) and $y'(0) = 0$ (the horizontal tangent).

7. Guessing solutions

Our main method for solving differential equations in this course will be: (drum roll...)

$$\boxed{\textbf{GUESSING!}}$$

Often we will guess the form of a solution. A suitable form for the solution will depend on a few parameters. We will adjust the parameters to make the solution work.

Example: consider the differential equation $y'' + 3y' - 4y = 0$. This equation ask us to find all functions $y(x)$ such that when you take a linear combination of $y(x)$ and two of its derivatives, you get zero. In other words, we are looking for a function $y(x)$ whose derivatives are very similar to it. One such function is $y = e^{ax}$, where a is a constant. Let's check to see if this works.

$$\begin{aligned}
& & y &= e^{ax} \\
&\Rightarrow & y' &= ae^{ax} \\
&\Rightarrow & y'' &= a^2 e^{ax} \\
&\Rightarrow & y'' + 3y' - 4y &= a^2 e^{ax} + 3ae^{ax} - 4e^{ax} \\
& & &= (a^2 + 3a - 4)e^{ax}.
\end{aligned}$$

So, this does not work... unless $a^2 + 3a - 4 = 0$, *i.e.*, $a = 1$ or $a = -4$. In other words,

$$y_1 = e^x, \quad y_2 = e^{-4x}$$

are both solutions. By guessing the functional form of a solution, we reduced the problem of solving a differential equation to the problem of solving an algebraic equation. This is definitely progress! We don't have all solutions yet, as the general solution should depend on two arbitrary constants. In the lectures on second-order equations we will learn how to use the two solutions we just found to construct the general solution.

8. Direction fields

For any first-order differential equation

$$y' = f(x, y),$$

we can get a graphical idea of what the solutions look like, even if we can't solve the equations. At any point (x_0, y_0) in the (x, y)-plane, the equation tells us what the rate of change of the solution through this point is. So, if we happen to find ourselves at this point (perhaps the initial condition put us there), the equation tells us how to move on from the point where we are. This is illustrated in Fig. 2.

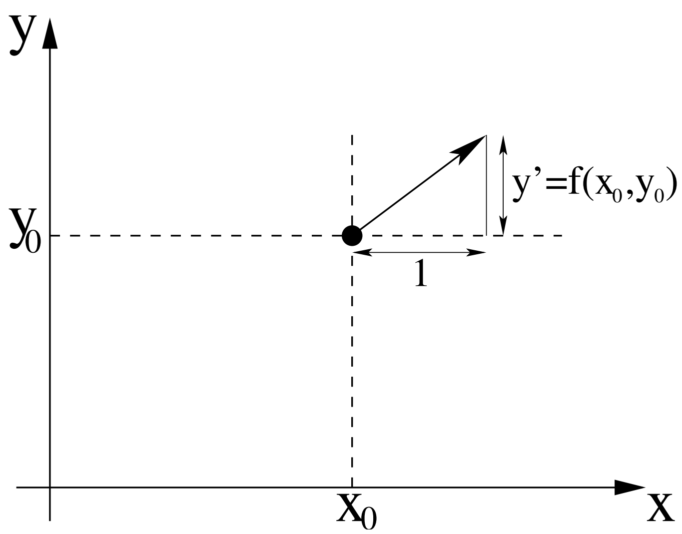

Figure 2: At any point (x_0, y_0) we may draw the tangent vector to the solution through this point.

The collection of all arrows through all points is called the **direction field** of the differential equation. The rate of change at any point gives the tangent vector to the solution curve through this point, allowing us to draw the tangent vector to the curve $y = y(x)$, which solves the differential equation, even if we cannot determine the form of this solution.

Thus, to find out what $y(x)$ looks like:

$$\boxed{\textbf{FOLLOW THE ARROWS!}}$$

Two examples of direction fields for two different differential equations are given in Figs. 3 and 4. Some solution curves are drawn as well.

As you may see from these direction fields, they may often be used to understand the long-time behavior of solutions, which in many applications is all we care about. For instance, in Fig. 4 it is clear that for all solutions with $y(0) > 1$ we have that $y(x) \to \infty$, as $x \to \infty$, and $y(x) \to -1$, as $x \to \infty$, if $y(0) < 1$. It appears that if $y(0) = 1$, then $y(x) = 1$, for all $x > 0$. This may easily be verified: plugging in allows us to verify immediately that $y(x) = 1$ is indeed a solution.

Of course, drawing all the tiny vectors in a direction field is a lot of work. It's also very boring work. In other words, it is the kind of work that a computer is very good at. On the course webpage you will find a link to a Java applet by John Polking and others to draw direction fields. It is available at `http://math.rice.edu/~dfield/dfpp.html`. The applet also allows you to draw in solution curves by clicking on the point through which you wish to draw a curve.

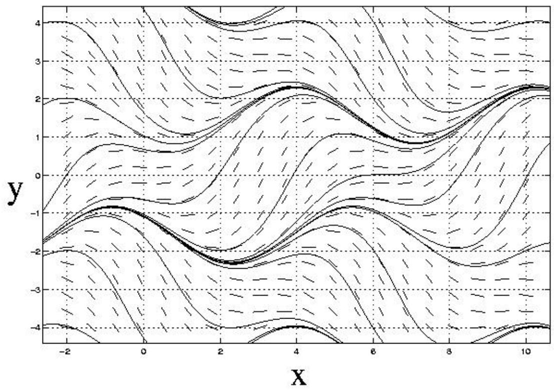

Figure 3: The direction field for the equation $y' = \cos(y) - \cos(x)$, together with some inferred solution curves.

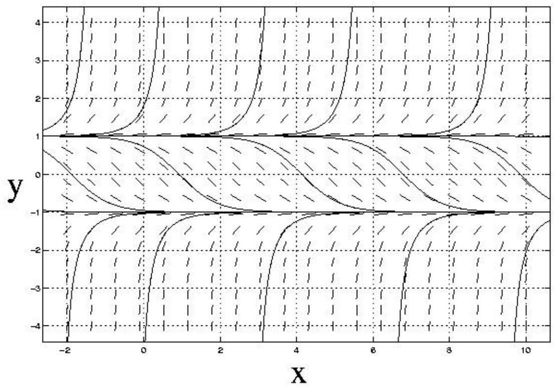

Figure 4: The direction field for the equation $y' = y^2 - 1$, together with some inferred solution curves.

Lecture 2. First-order separable differential equations

Almost all differential equations we will consider in this course are linear. This lecture is an exception.

Consider a differential equation of the form

$$\frac{dy}{dx} = f(x, y).$$

This is the most general form of a first-order differential equation. If $f(x, y)$ can be written as the product of a function of x and a function of y, we can solve the equation. Such an equation is called **separable**. It is of the form

$$\boxed{\frac{dy}{dx} = g(x)h(y).}$$

Then

$$\frac{1}{h(y)} y'(x) = g(x).$$

Integrating both sides with respect to x gives

$$\int \frac{1}{h(y)} y'(x) dx = \int g(x) dx.$$

The integral on the left may be rewritten as

$$\int \frac{1}{h(y)} dy,$$

and we get

$$\boxed{\int \frac{1}{h(y)} dy = \int g(x) dx + c,}$$

where we have written the constant of integration explicitly, so that we do not forget it.

Now the problem has been reduced to a calculus problem, and the differential equation has been solved. Note that even if we cannot do the integral, we consider the differential equation solved because there are no more derivatives in the problem.

Even if we can do the integral, it is unlikely that we can solve the resulting equation for y. So be it. If we can solve for y as a function of x we say that we have found an **explicit solution**. If not, we say we have an **implicit** solution.

Example: Consider the initial-value problem

$$\left\{ \begin{array}{rcl} y' & = & -6xy \\ y(0) & = & -4 \end{array} \right. .$$

From the first equation, we obtain

$$\begin{aligned}
\frac{1}{y}dy &= -6xdx \\
\Rightarrow \qquad \int \frac{1}{y}dy &= -6\int xdx + c \\
\Rightarrow \qquad \ln(y) &= -3x^2 + c.
\end{aligned}$$

This is an implicit solution of the differential equation. In this case, we can solve for y:

$$\begin{aligned}
y &= e^{-3x^2+c} \\
&= e^{-3x^2}e^c \\
&= Ce^{-3x^2},
\end{aligned}$$

where we have set $C = e^c$, an arbitrary constant. Now we may use the initial condition:

$$-4 = Ce^0 \quad \Rightarrow \quad C = -4,$$

and the explicit solution is

$$y = -4e^{-3x^2}.$$

Example: Consider

$$\begin{aligned}
y' &= \frac{3x^2+4x+2}{2(y-1)} \\
\Rightarrow \qquad (2y-2)dy &= (3x^2+4x+2)dx \\
\Rightarrow \qquad \int (2y-2)dy &= \int (3x^2+4x+2)dx + c \\
\Rightarrow \qquad y^2 - 2y &= x^3 + 2x^2 + 2x + c.
\end{aligned}$$

This is an implicit solution to the differential equation. In this case, we can actually write down the explicit solution. This amounts to solving the above quadratic equation for y explicitly, which may be done easily by completing the square:

$$\begin{aligned}
y^2 - 2y + 1 &= x^3 + 2x^2 + 2x + c + 1 \\
\Rightarrow \qquad (y-1)^2 &= x^3 + 2x^2 + 2x + c + 1 \\
\Rightarrow \qquad y - 1 &= \pm\sqrt{x^3 + 2x^2 + 2x + c + 1} \\
\Rightarrow \qquad y &= 1 \pm \sqrt{x^3 + 2x^2 + 2x + c + 1}.
\end{aligned}$$

This is the explicit solution of the differential equation. As you may deduce from this example, in many cases it is much harder to find an explicit solution than an implicit solution.

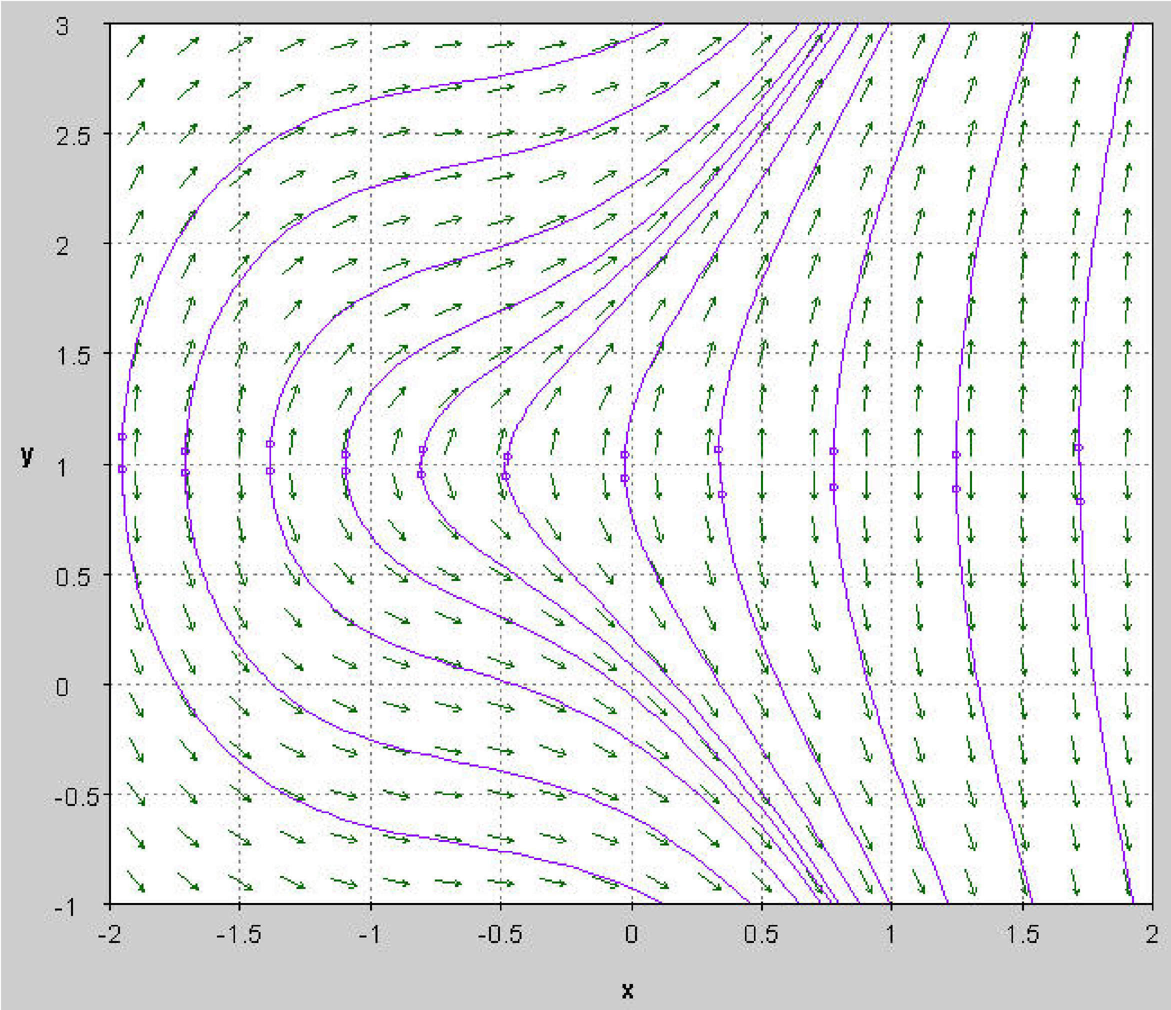

Figure 5: The solution curves for the differential equation $y' = (3x^2 + 4x + 2)/(2y - 2)$.

The solution curves for this example are plotted in Fig. 5. On the line $y = 1$, the solution curves have a vertical tangent. All curves above the line $y = 1$ correspond to the $+\sqrt{\ }$ for the explicit solution, whereas all curves below $y = 1$ correspond to the $-\sqrt{\ }$ for the explicit solution.

Example: Suppose we have to solve the initial-value problem

$$\begin{cases} y' &= \dfrac{3x^2 + 4x + 2}{2(y-1)} \\ y(0) &= -1 \end{cases} .$$

- Using the implicit solution, we get

$$1 + 2 = 0 + c \quad \Rightarrow \quad c = 3,$$

and thus

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3,$$

which tells us which solution curve to use, but not which part of it.

- Using the explicit solution we obtain

$$\begin{aligned} & -1 &=& \ 1 \pm \sqrt{c+1} \\ \Rightarrow \quad & -2 &=& \ \pm\sqrt{c+1}. \end{aligned}$$

  Independent of what value we find for c, this equality can only hold if we use the $-$ sign. Proceding with this choice:

$$-2 = -\sqrt{c+1} \quad \Rightarrow \quad \sqrt{c+1} = 2 \quad \Rightarrow \quad c = 3,$$

  giving the explicit solution

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

  The explicit solution conveys which solution curve has to be used, and also which part of it is found.

Example: This example demonstrates that it is not always possible to find an explicit solution, even if we can solve the differential equation. Consider the initial-value problem

$$\left\{ \begin{array}{rcl} \dfrac{dy}{dx} &=& \dfrac{y\cos x}{1+2y^2} \\ y(0) &=& 1 \end{array} \right. .$$

We obtain

$$\begin{aligned} & & \frac{1+2y^2}{y} dy &= \cos x dx \\ \Rightarrow & & \int \frac{1+2y^2}{y} dy &= \int \cos x dx + c \\ \Rightarrow & & \int \left(\frac{1}{y} + 2y\right) dy &= \sin x + c \\ \Rightarrow & & \ln y + y^2 &= \sin x + c. \end{aligned}$$

This is the implicit solution of the differential equation. It is not possible to solve this equation for y as a function of x, thus no explicit solution can be found. Nevertheless, we can still solve the initial-value problem. From the initial condition:

$$\ln 1 + 1^2 = \sin 0 + c \quad \Rightarrow \quad c = 1,$$

so that the implicit solution of the initial-value problem is

$$\ln y + y^2 = \sin x + 1.$$

In what follows, I want to demonstrate that when solving separable equations, you have to be careful when you divide. Consider the following

Example:

$$\begin{cases} y' &= y^2 \\ y(0) &= 0 \end{cases}.$$

- Proceeding without caution:

$$\begin{aligned} \Rightarrow \qquad & \frac{1}{y^2}dy = dx \\ \Rightarrow \qquad & \int \frac{1}{y^2}dy = \int dx + c \\ \Rightarrow \qquad & -\frac{1}{y} = x + c \\ \Rightarrow \qquad & y = \frac{1}{x+c}. \end{aligned}$$

  Now we use the initial condition, which leads to $0 = -1/c$, which cannot be solved for c! The problem occured right at the beginning, where we divided by y^2, which we may only do if $y \neq 0$. As it turns out, for the given differential equation, $y = 0$ is exactly what we need.

  In general, whenever we divide by a function of y, we need to check what happens when the denominator of this function is zero. Let's try this example again, being more careful.

- Proceeding with caution, we need to split the solution in two cases:
  - **Case $y = 0$.** In this case we cannot divide by y^2. Let's see if $y = 0$ is a solution of the differential equation: plugging in gives

    $$0 \stackrel{!}{=} 0,$$

    thus $y = 0$ is a solution! Even better, it is the solution that satisfies the initial condition. Thus, in summary, the solution of the initial value problem is $y = 0$.
  - **Case $y \neq 0$.** If different initial conditions are given, we have

    $$\begin{aligned} \Rightarrow \qquad & \frac{1}{y^2}dy = dx \\ \Rightarrow \qquad & \int \frac{1}{y^2}dy = \int dx + c \\ \Rightarrow \qquad & -\frac{1}{y} = x + c \\ \Rightarrow \qquad & y = \frac{1}{x+c}. \end{aligned}$$

Lecture 3. Linear first-order differential equations

In this lecture, we consider equations of the form

$$\boxed{y' + p(x)y = q(x),}$$

where $p(x)$ and $q(x)$ are given functions of x. We want to find all solutions $y(x)$.

Example: If $p(x) = 0$, this is easy:

$$y' = q(x) \quad \Rightarrow \quad y = \int q(x)dx + c.$$

Example: If $p(x) \neq 0$, it's not that easy. Consider the equation

$$y' + y = 5.$$

So here $p(x) = 1$, $q(x) = 5$. Let's rewrite this equation:

$$\begin{array}{lcl}
 & e^x(y' + y) = 5e^x & \\
\Rightarrow & e^x y' + e^x y = 5e^x & \\
\Rightarrow & (e^x y)' = 5e^x & \text{Using the product rule} \\
\Rightarrow & e^x y = 5e^x + c & \text{Integration} \\
\Rightarrow & y = e^{-x}(5e^x + c) & \\
\Rightarrow & y = 5 + ce^{-x}. &
\end{array}$$

This differential equation became easy to solve, once we multiplied it by the right function. So what is the right function? It is the function such that the left-hand side becomes the derivative of that function multiplied by y.

Let's see how we can accomplish this in general: we start with

$$y' + p(x)y = q(x).$$

Now we multiply by the right function, which we will call $\mu(x)$. At the moment we don't know this function yet. We will have to find a way to figure out what this function is, using the above requirement.

$$\begin{array}{lrcl}
 & \mu(x)\left(y' + p(x)y\right) & = & \mu(x)q(x) \\
\Rightarrow & \mu(x)y' + \mu(x)p(x)y & = & \mu(x)q(x).
\end{array}$$

Remember that we need the left-hand side to be a derivative. It should be $(\mu y)'$. Thus we need

$$\begin{array}{lc}
 & (\mu(x)y)' = \mu(x)y' + \mu(x)p(x)y \\
\Rightarrow & \mu' y + \mu y' = \mu y' + \mu p(x)y \\
\Rightarrow & \mu' y = \mu p(x)y \\
\Rightarrow & \mu' = \mu p(x).
\end{array}$$

This is a separable differential equation, like the ones we learned to solve in the previous lecture. Hurray! We get

$$\begin{aligned} & \mu' = \mu p(x) \\ \Rightarrow \quad & \frac{d\mu}{\mu} = p(x)dx \\ \Rightarrow \quad & \int \frac{d\mu}{\mu} = \int p(x)dx \\ \Rightarrow \quad & \ln \mu = \int p(x)dx \\ \Rightarrow \quad & \mu = e^{\int p(x)dx}. \end{aligned}$$

Note that we have ignored the constant of integration, as we only care about finding one function $\mu(x)$ for which the left-hand side becomes a derivative. There is no need to find all such functions. Once we have one such function $\mu(x)$, the differential equation becomes

$$(\mu(x)y)' = \mu(x)q(x).$$

Now we can integrate this, to get

$$\begin{aligned} \mu(x)y &= \int \mu(x)q(x)dx + c \\ \Rightarrow \quad y &= \frac{\int \mu(x)q(x)dx + c}{\mu(x)}. \end{aligned}$$

Let's summarize what we have found: To integrate a linear, first-order equation, we use the following steps:

0) Find $p(x)$, $q(x)$, *i.e.*, put the equation in the right form, ensuring that the coefficient of y' is 1.

1) Find $\boxed{\mu = e^{\int p(x)dx}}$.

2) $\boxed{y = \frac{\int \mu(x)q(x)dx + c}{\mu(x)}}$

3) If an initial condition is given, we can find c at this point.

Example: Let's try this on our original example:

$$y' + y = 5.$$

0) $p(x) = 1$, $q(x) = 5$.

1) $\mu = e^{\int p(x)dx} = e^{\int dx} = e^x$.

2) $y = \frac{\int \mu(x)q(x)dx + c}{\mu(x)} = \frac{\int e^x 5dx + c}{e^x} = \frac{5e^x + c}{e^5} = 5 + ce^{-x}$, which is the solution we found earlier.

3) No initial conditions are given, thus we are done.

Example: Let's do another one.

$$\begin{cases} y' + 2ty &= t, \\ y(0) &= 0. \end{cases}$$

0) $p(t) = 2t$, $q(t) = t$.

1) $\mu = e^{\int p(t)dt} = e^{\int 2tdt} = e^{t^2}$.

2) $y = \dfrac{\int \mu(t)q(t)dt + c}{\mu(t)} = \dfrac{\int e^{t^2} tdt + c}{e^{t^2}} = \dfrac{\frac{1}{2}e^{t^2} + c}{e^{t^2}} = \dfrac{1}{2} + ce^{-t^2}$.

3) Using the initial condition, $0 = \frac{1}{2} + c$, from which $c = -1/2$. The final solution to the initial-value problem is $y = \frac{1}{2}(1 - e^{-t^2})$.

Example: And maybe another one?

$$xy' + y = x$$

0) $p(x) = 1/x$, $q(x) = x/x = 1$, since we need to divide by x to ensure that y' has coefficient one. Otherwise our solution formulae are not valid.

1) $\mu = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x$.

2) $y = \dfrac{\int \mu(x)q(x)dx + c}{\mu(x)} = \dfrac{\int x * 1dx + c}{x} = \dfrac{x^2/2 + c}{x} = \dfrac{x}{2} + \dfrac{c}{x}$..

3) No initial conditions are given, thus we are done.

Lecture 4. Applications of first-order differential equations

Even simple differential equations like the ones we've seen so far have lots of interesting applications. In this lecture, we'll look at a few.

1. Mixture problems

Consider the set-up shown in Fig. 6. We need to introduce some names so we're all talking about the same thing. What are we looking at? On the left is a pipe, through which fluid is flowing into the big tank. This fluid has a certain concentration of a solvent, such as salt. On the right is the outlet pipe of the tank, through which the mixture is leaving. We're assuming that the concentration of the mixture is the same anywhere in the tank, so that the concentration of solvent in the mixture leaving the tank is the same as that of the concentration of solvent in the mixture in the tank.

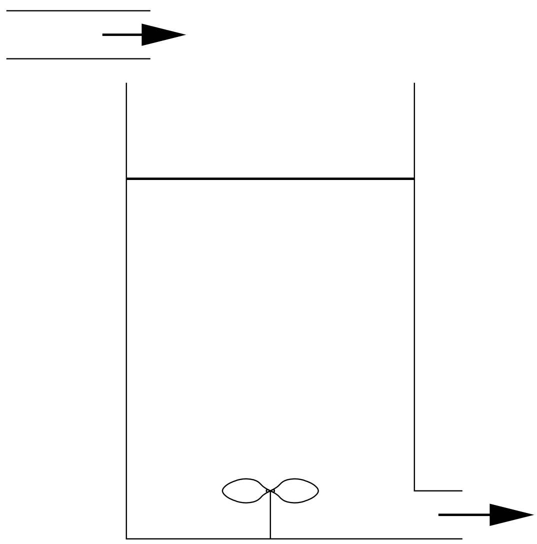

Figure 6: The set-up for a mixing problem

What names should we introduce?

- $m(t)$: the amount of solvent at time t in the tank. This will be in kilograms (kg).
- m_0: the starting amount of solvent $m(0)$, also in kg.
- $V(t)$: the volume of fluid mixture in the tank at time t (liters).
- V_0: the initial volume of fluid mixture $V(0)$ (liters).
- C_1: The concentration of solvent in the incoming fluid (kg/liters).
- C_2: The concentration of solvent in the outgoing fluid (kg/liters).
- q_1: the inflow rate (how fast is the fluid coming in; liters/sec).

- q_2: the outflow rate (how fast is the fluid going out; liters/sec).

Thus, the problem to be solved is to determine $m(t)$ for any time $t > 0$.

Here's the main differential equation to be used in lots of problems:

$$\boxed{\frac{dm}{dt} = \text{Increase in } m(t) \text{ per time unit} - \text{Decrease in } m(t) \text{ per time unit},}$$

which is really just the definition of what a derivative is. So, if we can figure out suitable expressions for the increase and the decrease of $m(t)$, we're all set.

-Increase per time unit: this is equal to how much is coming in to the tank*how fast it is coming in, or: increase of $m(t)$ per time unit $= C_1 q_1$.

-Decrease per time unit: similarly, this is equal to how much is going out of the tank*how fast it is going out, or: decrease of $m(t)$ per time unit $= C_2 q_2$.

Thus:

$$\frac{dm}{dt} = C_1 q_1 - C_2 q_2.$$

Some of the things in this equation we know: C_1, q_1 and q_2. The only one we don't know is C_2. Thus, so far we've got one equation, but it has two things we don't know. If we can determine C_2 in terms of things we know, or else in terms of the other thing we don't know $m(t)$, we're all set: then we'll have one equation with one unknown quantity. We'll solve the equation and have our answer! It's a cunning plan. So, how do we determine $C_2(t)$? Well,

$$C_2 = \text{amount of solvent in the tank/volume in the tank} = \frac{m(t)}{V(t)}.$$

But what is $V(t)$? Well, to determine $V(t)$, we can play the same game:

$$\begin{aligned}\frac{dV}{dt} &= \text{Increase in } V(t) \text{ per time unit} - \text{Decrease in } V(t) \text{ per time unit}\\ &= q_1 - q_2.\end{aligned}$$

This is a very easy differential equation to solve, especially if q_1 and q_2 are both constant. Then

$$V(t) = (q_1 - q_2)t + \alpha,$$

where α is an integration constant. To find α, we evaluate this expression for the volume at the only time at which we know something about the volume: at $t = 0$, $V(0) = V_0$. This given

$$V_0 = (q_1 - q_2)0 + \alpha = \alpha.$$

Thus, we find that the volume of mixture in the tank at any time is given by

$$V(t) = (q_1 - q_2)t + V_0.$$

We can already see a few interesting things from this equation: (i) if $q_1 > q_2$, more liquid is entering the tank than leaving it. The volume increases. (ii) if $q_1 < q_2$, more liquid is leaving the tank than entering it. The volume decreases. (iii) Lastly, if $q_1 = q_2$ there's as much going out as coming in, and the volume in the tank stays the same.

Now we use the expression we just found in the differential equation for $m(t)$, to get

$$\boxed{\frac{dm}{dt} = C_1 q_1 - q_2 \frac{m}{(q_1 - q_2)t + V_0}.}$$

This is a linear, first-order differential equation of the kind we learned how to solve in the previous lecture. What a coincidence! In order to solve it, we first rewrite the equation in the form we had in the previous lecture:

$$\frac{dm}{dt} + \frac{q_2}{(q_1 - q_2)t + V_0} m = C_1 q_1.$$

Using the steps from the previous lecture, we have:

0) $p = \dfrac{q_2}{(q_1 - q_2)t + V_0}$, $q = C_1 q_1$.

1) $\mu = e^{\int p dt} = e^{\int \frac{q_2}{(q_1 - q_2)t + V_0} dt}$. You see there are two cases:

   (a) $q_1 = q_2$ (the case of constant volume). In this case there's no t-dependence in the denominator, and we get $\mu = e^{q_2 t / V_0}$.

   (b) $q_1 \neq q_2$ (changing volume). In this case, the integration is a bit more complicated:

$$\begin{aligned} \mu &= e^{\frac{q_2}{q_1 - q_2} \ln(V_0 + (q_1 - q_2)t)} \\ &= e^{\ln(V_0 + (q_1 - q_2)t)^{q_2/(q_1 - q_2)}} \\ &= (V_0 + (q_1 - q_2)t)^{q_2/(q_1 - q_2)}. \end{aligned}$$

   Note that this equation only makes sense as long as the volume is positive: if the volume is decreasing, it will become zero at some point. After that, this calculation stops making sense.

Let's proceed with Case (a), which is more important for applications.

2) The solution is given by

$$\begin{aligned} m &= \frac{\int \mu q dt + c}{\mu} \\ &= \frac{\int e^{q_2 t / V_0} C_1 q_1 dt + c}{e^{q_2 t / V_0}} \\ &= \frac{q_1 C_1 e^{q_2 t / V_0} \frac{V_0}{q_2} + c}{e^{q_2 t / V_0}} \\ &= C_1 V_0 \frac{q_1}{q_2} + c e^{-q_2 t / V_0}. \end{aligned}$$

Since $q_1 = q_2$, this simplifies to

$$m = C_1 V_0 + c e^{-q_2 t/V_0}.$$

3) At $t = 0$, this equation gives

$$m_0 = C_1 V_0 + c \;\Rightarrow\; c = m_0 - C_1 V_0,$$

so that, finally, the amount of solvent in the tank at any time t is

$$\boxed{m = C_1 V_0 + (m_0 - C_1 V_0) e^{-q_2 t/V_0}.}$$

Having obtained this result, we can analyze it to obtain interesting results. Also, we should check that the outcome agrees with the intuition we have about the problem. For instance, since we keep on pouring in concentration C_1, it seems reasonable that the eventual concentration $m(t)/V(t)$ should approach C_1. Let's see:

$$\begin{aligned}\lim_{t\to\infty} \frac{m}{V} &= \lim_{t\to\infty} C_1 + \left(\frac{m_0}{V_0} - C_1\right) e^{-q_2 t/V_0} \\ &= C_1,\end{aligned}$$

as expected.

2. Some examples from mechanics

In mechanics, you have seen a great many formulas relating quantities such as velocity, acceleration, position, *etc.* Whole collections of such results exist, and their derivation relies on distinct arguments in different settings. We'll show now that all such formulas all follow from one differential equation, namely, **Newton's law**. Newton's law equates force with mass times accelerations:

$$F = ma,$$

where F denotes the force acting on a particle, m is its mass, and a is its acceleration. How is this even a differential equation? Well, we know that the velocity is the derivative of the position s, and the acceleration is the derivative of the velocity v. This is how the derivatives come in. The general principle of dynamics is to specify forces. Once this is done, Newton's law gives us acceleration. Since the forces may depend on velocities and position, we typically have a second-order differential equation for the position as a function of time.

If we look at the important case of constant acceleration, we have

$$a = \frac{dv}{dt} \quad\Rightarrow\quad v = at + c,$$

since we're assuming that a is constant. Evaluating this at $t = 0$, when the initial velocity is $v(0) = v_0$ gives

$$v_0 = c \quad\Rightarrow\quad v = v_0 + at.$$

We've integrated once: we won't be stopped. Let's put in that the velocity is the derivative of the position. This gives:

$$\frac{ds}{dt} = v_0 + at \quad \Rightarrow \quad s = \hat{c} + v_0 t + \frac{1}{2}at^2.$$

At $t = 0$, with $s(0) = s_0$, we get

$$s_0 = \hat{c} \quad \Rightarrow \quad \boxed{s = s_0 + v_0 t + \frac{1}{2}at^2},$$

which may look familiar. Note how systematically these results were derived using even the simplest differential equations!

An important special case of constant acceleration is that of a body falling under the influence of gravity close to the surface of the Earth. Then

$$a = -g,$$

where we choose the position axis to point upward, hence gravity is represented by a negative acceleration. Our previous results become

$$\begin{aligned} v &= v_0 - gt, \\ s &= s_0 + v_0 t - \frac{1}{2}t^2. \end{aligned}$$

If the body is falling from rest from height h, then $s_0 = h$ and $v_0 = 0$. How long does it take the body to fall? We have

$$s = h - \frac{1}{2}gt^2.$$

This is zero when

$$h = \frac{1}{2}gt^2 \quad \Rightarrow \quad t^2 = \frac{2h}{g} \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}}.$$

The velocity at the time of impact is

$$v = -gt \quad \Rightarrow \quad v = -\sqrt{2gh}.$$

Both results probably look familiar.

Radioactive decay

Radioactive decay is a process through which an isotope of an element transforms into another isotope of the same element. Typically, a sample of any species will be a mixture of different isotopes. By measuring how much of a certain isotope has decayed, we can determine the age of the sample, for instance. This is the basis for the method of carbon dating.

Radioactive decay is governed by a simple first-order differential equation

$$\boxed{\frac{dN}{dt} = -\kappa N.}$$

This equation states that the rate of change of the amount of isotope N is decaying (the minus sign on the right) proportional to the amount of the isotope: if there's a lot of the isotope, lots will decay. If there's only a little, only a small amount will decay. The constant κ is known as the decay constant. The solution of this simple differential equation is (check this, or even better: find it!)

$$N = N_0 e^{-\kappa t},$$

where $N_0 = N(0)$ is the initial amount of isotope.

The **half-life time** τ is the time it takes for half of the amount of isotope to decay. Let's figure out what it is. At the half-life time $t = \tau$ and $N = N_0/2$. This gives

$$\begin{aligned}
& & \frac{N_0}{2} &= N_0 e^{-\kappa\tau} \\
&\Rightarrow & \ln\frac{1}{2} &= -\kappa\tau \\
&\Rightarrow & -\ln 2 &= -\kappa\tau \\
&\Rightarrow & & \boxed{\tau = \frac{1}{\kappa}\ln 2.}
\end{aligned}$$

Determining the age of the Universe: assuming there was an equal amount of U_{235} and U_{238} (two Uranium isotopes) at the Big Bang, and that currently there are 137.7 U_{238} for every U_{235}, how old is the Universe? We also know the half-life times for both isotopes: $\tau_{238} = 4.5\,10^9$yrs and $\tau_{235} = 7\,10^8$yrs.

Since $N_{0,238} = N_{0,235} = N_0$, we know

$$\begin{aligned}
& & N_{238} &= N_0 e^{-\frac{t}{\tau_{238}}\ln 2} \\
& & N_{235} &= N_0 e^{-\frac{t}{\tau_{235}}\ln 2} \\
&\Rightarrow & \frac{N_{238}}{N_{235}} &= \frac{N_0 e^{-\frac{t}{\tau_{238}}\ln 2}}{N_0 e^{-\frac{t}{\tau_{235}}\ln 2}} \\
&\Rightarrow & 137.7 &= e^{\frac{t}{\tau_{235}}\ln 2 - \frac{t}{\tau_{238}}\ln 2} \\
&\Rightarrow & \ln 137.7 &= t\ln 2\left(\frac{1}{\tau_{235}} - \frac{1}{\tau_{238}}\right) \\
&\Rightarrow & t\frac{\tau_{238} - \tau_{235}}{\tau_{235}\tau_{238}} &= \frac{\ln 137.7}{\ln 2} \\
&\Rightarrow & t &= \frac{\ln 137.7}{\ln 2}\frac{\tau_{235}\tau_{238}}{\tau_{238} - \tau_{235}} \\
&\Rightarrow & t &= 0.6\ 10^{10}\text{yrs},
\end{aligned}$$

which is quite old.

Lecture 5. Stability and phase plane analysis

Let's look at another application. This one comes from population dynamics.

1. Linear model

Suppose we're examining the growth of a small population, with plenty of resources and no predators. Denote this population by $y(t)$. Then, it seems reasonable that

$$\frac{dy}{dt} = \alpha y,$$

where $\alpha > 0$ is the growth constant: the population increases over time. We're stating that the population growth is proportional to the size of the population. Solving this equation, as before, gives

$$y = y_0 e^{\alpha t},$$

and the population grows exponentially with time. We've used $y_0 = y(0)$ to denote the initial population. This is reasonable, with the assumptions we've put in place. However, the above solution says that the population will experience very rapid growth. Eventually, maybe after a long time, the amount of resources might not be sufficient to support this growth. Now what happens? To include effects like this, we need a nonlinear model.

2. Nonlinear model

We'll modify our model as follows:

$$\frac{dy}{dt} = \alpha \left(1 - \frac{y}{K}\right) y = \alpha y - \frac{\alpha}{K} y^2.$$

Using the first formulation, we may think of $\alpha(1-y/K)$ as our growth constant, as for the linear model. Now, the growth constant is dependent on y (so, it's not really a constant anymore): for $y < K$, it's positive and it looks like the population will grow. On the other hand, for $y > K$, the growth constant is negative and the population will decrease. Using the second formulation, we can think of the right-hand side of the differential equation as the first two terms of a Taylor expansion.

Let's solve this new, nonlinear differential equation. Note that it is a separable equation:

$$\begin{aligned}
& & \frac{dy}{dt} &= \alpha \left(1 - \frac{y}{K}\right) y \\
&\Rightarrow & \int \frac{dy}{\left(1 - \frac{y}{K}\right) y} &= \alpha \int dt \\
&\Rightarrow & \int \left(\frac{1}{y} + \frac{1}{K}\frac{1}{1 - \frac{y}{K}}\right) dy &= \alpha t + c \\
&\Rightarrow & \ln|y| - \ln\left|1 - \frac{y}{K}\right| &= \alpha t + c,
\end{aligned}$$

where we have used partial fractions. After some more algebra, we get (check this!)

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-\alpha t}},$$

with $y_0 = y(0)$.

Let's examine this solution to find out what happens to the population as $t \to \infty$: does it die out? Does it persist? Does it grow forever?

- If $y_0 = 0$, then $\lim_{t\to\infty} y = 0$: if there's no population to start with, things aren't very interesting.
- If $y_0 = K$, then $\lim_{t\to\infty} y = K$: apparently K is that population level that is in perfect balance with its surroundings!
- If $0 < y_0 < K$, then $\lim_{t\to\infty} y = K$: if we start with a small population (*i.e.*, less than K), the population grows towards the balance population.
- If $y_0 > K$, then $\lim_{t\to\infty} y = K$: if we start with a population that is too large to be sustained by the available resources, the population decreases towards the balance population.

The information above is often the main information we are interested in obtaining: what will happen with the solutions if we wait long enough? And how does this depend on the initial conditions? Gee, that was quite a bit of work to get that information! Is there an easier way to get it?

There is: let's look at the differential equation again:

$$y' = \alpha \left(1 - \frac{y}{K}\right) y.$$

Without solving this equation, we can draw the **Phase-line picture**: we plot y' as a function of y: in other words we plot the right-hand side of the differential equation. This is done in Fig. 7.

This simple figure tells us a lot. It shows that whenever y is between 0 and K, the graph is positive, which means that $y' > 0$. This implies that y will increase. Hence, if we start somewhere on this interval, with a certain y value, we'll move to y-values that are more to the right, until we reach $y = K$. At that point, we stop because there the graph has a zero, which implies $y' = 0$, thus y does not change anymore. Similarly, if y starts off to the right of $y = K$, the graph is negative, which means that $y' < 0$, y will decrease: we'll move to the left, again until we reach $y = K$. We could also investigate what happens for $y < 0$, but such populations are not very interesting.

The two values $y = 0$ and $y = K$ stand out, because for these values the graph is zero, thus $y' = 0$, and there is no change once these values are attained. At these y-values there is no change. Such values are called **fixed points**, **equilibrium solutions**, or **equilibrium points**.

In general, an **equilibrium point** of a differential equation is a constant solution of a differential equation, such that $y' = 0$.

Here's some more defitions:

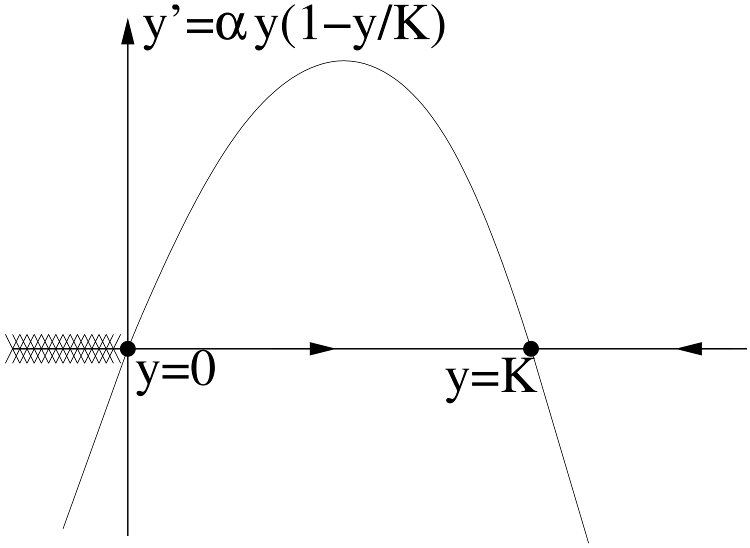

Figure 7: The phase-line plot for the population growth model

- An equilibrium point is called **asymptotically stable** if solutions close to it, get closer to it.
- An equilibrium point is called **unstable** if solutions close to it get further away.
- An equilibrium point is called **semi-stable** if some nearby solutions get further away, and if others get closer.

The **phaseline plot** contains all this information in a very concise way. For the previous population example, a more compact version of the phase line plot is shown in Fig. 8. We can omit the plot of the right-hand side of the differential equation: what really matters is where that right-hand side is zero. These points are the equilibrium points. They are indicated in Fig. 8. Once we have these points, all that's left to do is to see how the values in between these equilibrium points change. This is indicated using a left or right arrow. If these arrows point away from an equilibrium point, that equilibrium

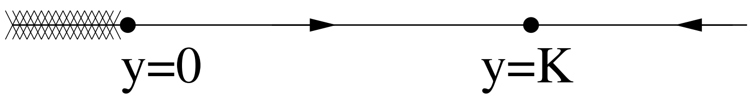

Figure 8: The compact phase-line plot for the population growth model

point is unstable. If all arrows point towards the equilibrium point, it is asymptotically stable. Otherwise it is semi-stable. In our example, we have

- $y_1 = 0$ is an unstable equilibrium point.
- $y_2 = K$ is an asymptotically stable equilibrium point.

Let's do another example:

Example: Consider the differential equation $y' = \alpha(1-y)^2 y(-2+y)$, where α is a positive constant. If we'd be asked to solve this equation, we're in for quite a bit of work. On the other hand if we're asked to determine the equilibrium solutions and their stability, this is quite easy.

First, we find there are three equilibrium points, namely those y-values that make the right-hand side zero. Thus $y_1 = 0$, $y_2 = 1$, and $y_3 = 2$ are equilibrium points. Now we can draw the phase-line plot. It and its compact version are shown in Fig. 9.

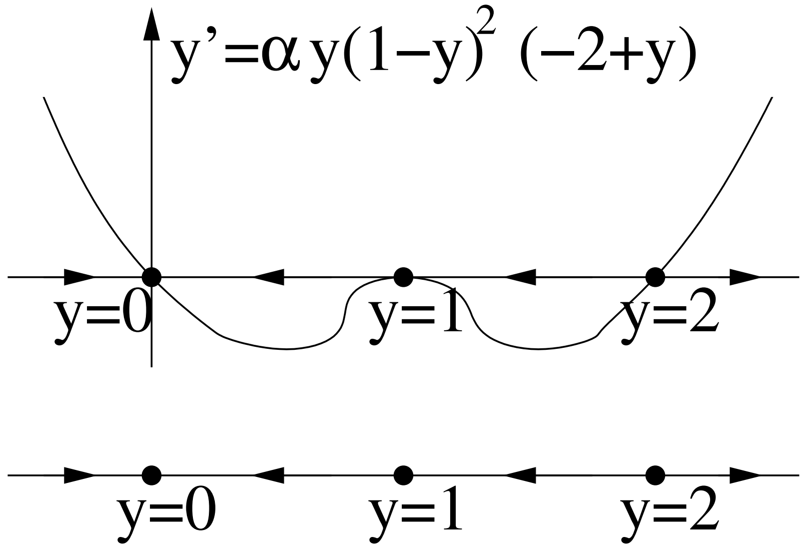

Figure 9: The phase-line plot for $y' = \alpha(1-y)^2 y(-2+y)$.

From these plots, we obtain immediately that:

- $y_1 = 0$ is an asymptotically stable equilibrium point.
- $y_2 = 1$ is a semi-stable equilibrium point.
- $y_3 = 2$ is an unstable equilibrium point.

Lecture 6. Exact differential equations

Suppose we have a function $y(x)$, which is defined by

$$f(x,y) = x^2 + xy^2 = c,$$

where c is a constant. We could solve this for y, but that would require square roots, so let's not. Can we figure out what differential equation y satisfies? Let's take a derivative of our equation:

$$\begin{aligned} & \frac{d}{dx}\left(x^2 + xy^2\right) = 0 \\ \Rightarrow \qquad & 2x + y^2 + 2xy\frac{dy}{dx} = 0, \end{aligned}$$

where we have used the chain rule. Thus, our function y satisfies the differential equation

$$2xyy' + 2x + y^2 = 0.$$

Now, this is a pretty tough looking equation: it's nonlinear, and not separable. But we know its solution, since that's what we started from. It would be a shame if there wasn't a method to solve for it. So, since the world is a nice place and we are all enjoying eternal bliss, there is a method to solve differential equations like the one above.

Let's look at this problem in more generality. Suppose we have a function y defined by the implicit equation

$$f(x,y) = c,$$

where c is a constant. Taking a derivative, using the chain rule we get

$$\begin{aligned} & \frac{d}{dx}f(x,y) = 0 \\ \Rightarrow \qquad & \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \\ & \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial x} = 0. \end{aligned}$$

Thus we have an equation of the form

$$N(x,y)y' + M(x,y) = 0.$$

We know its solution if the following equations hold:

$$\boxed{\left\{\begin{array}{rcl} M(x,y) & = & \dfrac{\partial f}{\partial x}, \\ N(x,y) & = & \dfrac{\partial f}{\partial y}. \end{array}\right.} \tag{1}$$

In that case the solution is $f(x,y) = c$. So how do we know if these statements hold? Well, suppose they do, then certainly by the equality of mixed derivatives

$$\begin{aligned} & \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \\ \Rightarrow \qquad & \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.} \end{aligned} \tag{2}$$

This is an easy equation to test, given a differential equation of the form

$$N(x,y)y' + M(x,y) = 0.$$

If the equality (2) holds, the differential equation is called **exact**, and the equations (1) will have a solution for $f(x,y)$ so that the general solution of the equation is

$$\boxed{f(x,y) = c.}$$

Let's see how we can use this to solve the problem we started with.

Example: Consider the differential equation $2xyy' + 2x + y^2 = 0$. Then

$$M = 2x + y^2, \quad N = 2xy.$$

First we check if the equation is exact:

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2y.$$

Since these are equal, the equation is exact. Now we can proceed to solve (1):

$$\begin{cases} M(x,y) &= \dfrac{\partial f}{\partial x} = 2x + y^2, \\ N(x,y) &= \dfrac{\partial f}{\partial y} = 2xy. \end{cases}$$

We can solve these equations in the order we prefer. Let's start with the first equation. Then

$$\begin{aligned} & \frac{\partial f}{\partial x} = 2x + y^2 \\ \Rightarrow \qquad & f = \int (2x + y^2)\partial x + h(y). \end{aligned}$$

I've used the notation ∂x to denote that this is an integration with respect to x, where we are thinking of y as a constant. This is why the constant of integration $h(y)$ could possibly depend on it. Proceeding,

$$f = x^2 + y^2 x + h(y).$$

We now substitute this in the second equation. This gives

$$\begin{aligned} & \frac{\partial f}{\partial y} = 2xy + h'(y) = 2xy \\ \Rightarrow \qquad & h'(y) = 0 \\ \Rightarrow \qquad & h(y) = 0. \end{aligned}$$

Note that we can let $h(y) = 0$, instead of $h(y) = c$, since we're already incorporating our integration constant in our general solution $f(x,y) = c$. Now we can write down what we found for $f(x,y)$, to write the solution of the differential equation:

$$f(x,y) = c \quad \Rightarrow \quad x^2 + y^2 x = c,$$

as expected.

Example: Let's do a more complicated example. Consider

$$(\sin x + x^2 e^y - 1)y' + (y\cos x + 2xe^y) = 0,$$

a nonlinear differential equation if ever there was one. Here

$$M = y\cos x + 2xe^y, \quad N = \sin x + x^2 e^y - 1.$$

Let's check if this differential equation is exact:

$$\frac{\partial M}{\partial y} = \cos x + 2xe^y, \quad \frac{\partial N}{\partial x} = \cos x + 2xe^y.$$

These are equal, thus the equation is exact. Thus

$$\begin{cases} M(x,y) &= \dfrac{\partial f}{\partial x} = y\cos x + 2xe^y, \\ \\ N(x,y) &= \dfrac{\partial f}{\partial y} = \sin x + x^2 e^y - 1. \end{cases}$$

Using the first equation:

$$\begin{aligned} f &= \int (y\cos x + 2xe^y)\, \partial_x + h(y) \\ &= y\sin x + x^2 e^y + h(y). \end{aligned}$$

Then $\partial f/\partial y = \sin x + x^2 e^y + h'(y)$. Plugging this in the second equation gives

$$\begin{aligned} & \sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y - 1 \\ \Rightarrow \quad & h'(y) = -1 \\ \Rightarrow \quad & h(y) = -y. \end{aligned}$$

Our final solution is

$$f(x,y) = y\sin x + x^2 e^y - y = c.$$

Example: Consider

$$(x^2 + xy)y' + 3xy + y^2 = 0.$$

Here we have

$$M = 3xy + y^2, \quad N = x^2 + xy.$$

Let's check if this equation is exact:

$$\frac{\partial M}{\partial y} = 3x + 2y, \quad \frac{\partial N}{\partial x} = 2x + y.$$

Since these are not equal, the equation is **not exact**. What would happen if we ignored this and proceded anyways? Let's give it a try. Then

$$\begin{cases} M(x,y) &= \dfrac{\partial f}{\partial x} = 3xy + y^2, \\ \\ N(x,y) &= \dfrac{\partial f}{\partial y} = x^2 + xy. \end{cases}$$

From the first equation we get

$$f = \int \left(3xy + y^2\right) \partial x + h(y) = \frac{3}{2}x^2 y + y^2 x + h(y).$$

Substituting this in the second equation to determine $h(y)$ gives

$$\begin{aligned} & \frac{3}{2}x^2 + 2xy + h'(y) = x^2 + xy \\ \Rightarrow \qquad & h'(y) = -\frac{1}{2}x^2 - xy. \end{aligned}$$

This equation cannot be solved, since $h(y)$ is only allowed to depend on y, not on x. This happened because the equation is not exact. The exactness condition guarantees that terms will cancel so that the function we have to determine after having done one integration only depends on the remaining variable. So, we're stuck. The equation is not exact, and we do not know a way of solving it.

Or do we? Let's take the same equation, but multiply it by x:

$$\begin{aligned} & x\left((x^2 + xy)y' + 3xy + y^2\right) = 0 \\ \Rightarrow \qquad & (x^3 + x^2 y)y' + 3x^2 y + xy^2 = 0. \end{aligned}$$

Clearly this equation has the same solutions as the equation we were trying to solve originally. But now M and N are different:

$$M = 3x^2 y + xy^2, \quad N = x^3 + x^2 y.$$

Would, by chance, this equation be exact? Let's try:

$$\frac{\partial M}{\partial y} = 3x^2 + 2xy, \quad \frac{\partial N}{\partial x} = 3x^2 + 2xy.$$

These are equal, and the equation is exact! Sweet. Let's solve it.

$$\begin{cases} M(x,y) &= \dfrac{\partial f}{\partial x} = 3x^2 y + xy^2, \\ \\ N(x,y) &= \dfrac{\partial f}{\partial y} = x^3 + x^2 y. \end{cases}$$

Just for the heck of it, let's solve the second equation first: (try as an exercise to do it with the first equation first)

$$\begin{aligned} f &= \int \left(x^3 + x^2 y\right) \partial y + h(x) \\ &= x^3 y + \frac{1}{2}x^2 y^2 + h(x). \end{aligned}$$

Plugging this in the first equation gives

$$\begin{aligned} & 3x^2 y + xy^2 + h'(x) = 3x^2 y + xy^2 \\ \Rightarrow \qquad & h'(x) = 0 \\ \Rightarrow \qquad & h(x) = 0. \end{aligned}$$

Finally, the solution is:

$$f(x,y) = x^3 y + \frac{1}{2}x^2 y^2 = c.$$

Lecture 7. Substitutions for first-order differential equations

Sometimes integrals become a lot simpler to evaluate if you use the right substitution. The same is true for differential equations. Unfortunately, just like for integrals, it takes experience to get a feel for what the "right" substitution is.

In this lecture, you'll learn how to use a substitution on a given differential equation, but I don't expect you to come up with the substitution to use. The ability to do this will come in time and by doing an unreasonable amount of homework problems.

Example: Consider the differential equation

$$y' = \frac{y^2 + 2xy}{x^2}.$$

This equation is nonlinear, non-separable and not exact. (verify this!) Let's rewrite it:

$$y' = \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right).$$

This suggests that the differential equation might be simpler in terms of the function $u(x) = y(x)/x$. In order to see whether this is true, let's find a differential equation that $u(x)$ satisfies. We're hoping this will be a simpler differential equation than the one we're starting with. If not, we'll have to try something else.

Using a substitution on a differential equation proceeds in four steps:

1. **Write down the substitution and the inverse substitution**: here

$$u(x) = \frac{y(x)}{x}, \qquad y(x) = xu(x).$$

2. **Take a derivative of the new function**:

$$u'(x) = \frac{xy'(x) - y(x)}{x^2},$$

   where we have used the chain rule.

3. **Replace y' using the original differential equation**:

$$\begin{aligned} u' &= \frac{xy' - y}{x^2} \\ &= \frac{x\frac{y^2+2xy}{x^2} - y}{x^2} \\ &= \frac{\frac{y^2}{x} + 2y - y}{x^2} \\ &= \frac{y^2/x + y}{x^2} \\ &= \frac{y^2}{x^3} + \frac{y}{x}. \end{aligned}$$

4. **Replace y, using step 1**:

$$\begin{aligned} u' &= \frac{y^2}{x^3} + \frac{y}{x} \\ &= \frac{x^2u^2}{x^3} + \frac{xu}{x^2} \\ &= \frac{u^2}{x} + \frac{u}{x} \\ &= \frac{u^2+u}{x}. \end{aligned}$$

Now we have a new differential equation, expressing u' in terms of x and u. This new equation is separable, so we proceed to solve it. There's two cases.

- Case $u^2 + u = 0$. Then we cannot divide by $u^2 + u$. This happens if $u = 0$ or $u = -1$. This corresponds to $y = 0$ or $y = -x$. You can easily check that these are indeed solutions of the original differential equation.
- Case $u^2 + u \neq 0$. Then

$$\begin{aligned} \int \frac{1}{u(u+1)} du &= \int \frac{1}{x} dx + c \\ &= \ln x + c \\ &= \ln x + \ln \hat{c} \\ &= \ln(x\hat{c}). \end{aligned}$$

To do the integral on the left we need partial fractions:

$$\begin{aligned} && \frac{1}{u(u+1)} &= \frac{A}{u} + \frac{B}{u+1} \\ \Rightarrow && 1 &= A(u+1) + Bu \\ \Rightarrow && A + B = 0, &\quad A = 1 \\ \Rightarrow && B = -1, &\quad A = 1. \end{aligned}$$

We get

$$\begin{aligned} \ln(x\hat{c}) &= \int \left(\frac{1}{u} - \frac{1}{u+1}\right) dx \\ &= \ln u - \ln(u+1) \\ &= \ln \frac{u}{u+1}. \end{aligned}$$

This gives

$$\begin{aligned} && \frac{u}{u+1} &= \hat{c}x \\ \Rightarrow && u &= \hat{c}x(u+1) \\ \Rightarrow && u(1-\hat{c}x) &= \hat{c}x \\ \Rightarrow && u &= \frac{\hat{c}x}{1-\hat{c}x} \end{aligned}$$

In terms of y, we get

$$\frac{y}{x} = \frac{\hat{c}x}{1-\hat{c}x} \quad \Rightarrow \quad y = \frac{\hat{c}x^2}{1-\hat{c}x}.$$

Thus all solutions are

$$y = 0, \quad y = -x, \quad y = \frac{\hat{c}x^2}{1-\hat{c}x}.$$

Note that if we were unable to solve for y at the end, we'd still be able to find an implicit solution for y.

Let's summarize the steps we'll use to work with a substitution $u = u(x,y) = u(x,y(x))$ and a differential equation $y' = f(x,y)$:

1. The inverse substitution: $y = y(x,u)$.
2. Take a derivative of the new variable: $u' = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} y'$, by the chain rule.
3. Use the differential equation: $u' = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} f(x,y) = F(x,y,u)$.
4. Use the inverse substitution: $u' = F(x, y(x,u), u) = G(x,u)$.

This all leads a new differential equation $u' = G(x,u)$ for the new function.

Example: Consider the equation

$$x^2 y' + 2xy = y^3,$$

which is (check this!) nonlinear, non-separable and not exact. We'll solve it using the substitution $v = 1/y^2$. Note that by using this substitution we are excluding the solution $y = 0$, which we have to add in separately.

1. The inverse substitution: $y = \pm v^{-1/2}$.
2. Take a derivative of the new variable: $v' = -\frac{2}{y^3} y'$, by the chain rule.
3. Use the differential equation: $v' = -\frac{2}{y^3} \frac{y^3 - 2xy}{x^2} = \frac{-2 + 4x\frac{1}{y^2}}{x^2}$.
4. Use the inverse substitution: $v' = \frac{-2 + 4xv}{x^2}$.

We have found a new differential equation for the function v, which is indeed simpler than the original equation: the new equation is linear!

$$v' - \frac{4}{x} v = -\frac{2}{x^2}.$$

Step 0. $p = -4/x$, $q = -2/x^2$

Step 1. $\mu = e^{\int p dx} = e^{\int \frac{-4}{x} dx} = e^{-4 \ln x} = \dfrac{1}{x^4}$.

Step 2. $v = \dfrac{\int \mu q dx + c}{\mu} = \dfrac{\int x^{-4}(-2)x^{-2} dx + c}{x^{-4}} = x^4 \left(-2 \int x^{-6} dx + c \right) = x^4 \left(\dfrac{2}{5} x^{-5} + c \right) = \dfrac{2}{5} x^{-1} + c x^4 = \dfrac{2 + c x^5}{5x}$.

Thus $y = \pm v^{-1/2} = \pm \sqrt{\dfrac{5x}{2 + cx^5}}$. This, together with $y = 0$, provides the general solution to the differential equation. On a sidenote, the original differential equation may also be rewritten as $(x^2 y)' = y^3$, which leads to another substitution to solve this equation (try it).

Example: Let's look at one more example:

$$y' = 1 + x^2 - 2xy + y^2 = 1 + (y - x)^2.$$

This second form of the equation suggests that perhaps $u = y - x$ is a good idea for a substitution. Let's try it.

1. The inverse substitution: $y = u + x$.
2. Take a derivative of the new variable: $u' = y' - 1$.
3. Use the differential equation: $u' = 1 + (y - x)^2 - 1 = (y - x)^2$.
4. Use the inverse substitution: $u' = u^2$.

This equation is separable:

$$\int \frac{1}{u^2} du = \int dx + c \Rightarrow -\frac{1}{u} = x + c \Rightarrow u = -\frac{1}{x + c},$$

so that

$$y = x - \frac{1}{x + c}.$$

Lecture 8. Second-order, constant-coefficient equations

In this lecture, we'll look at second-order equations for the first time. All the second-order equations we'll consider here will be linear. We won't look at nonlinear equations again until we get to nonlinear systems, much later in these notes.

Any second-order linear equation is of the form

$$r(x)y'' + p(x)y' + q(x)y = g(x),$$

where $r(x)$, $p(x)$ and $q(x)$ may be functions of x. For now, we'll assume they are constants: $r(x) = a$, $p(x) = b$ and $q(x) = c$, with a, b and c constant. Further, we'll start with the **homogeneous case**, *i.e.*, the case where $g(x) = 0$. Thus, we'll consider

$$ay'' + by' + cy = 0,$$

where $y = y(x)$ is the function we're looking for. Let's introduce some shorthand. Let

$$L[y] = ay'' + by' + cy,$$

so that the differential equation simply is $L[y] = 0$. Let's discuss a few properties of this equation.

Theorem 1 (Principle of Superposition) *If y_1 and y_2 are independent solutions of this equation, then $y(x) = c_1y_1(x) + c_2y_2(x)$ is the general solution.*

Proof: Note that the general solution will depend on two constants, since we are now dealing with a second-order equation.

$$\begin{aligned}
L[y] &= ay'' + by' + cy \\
&= a(c_1y_1'' + c_2y_2'') + b(c_1y_1' + c_2y_2') + (c_1y_1 + c_2y_2) \\
&= c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2) \\
&= c_1L[y_1] + c_2L[y_2] \\
&= c_10 + c_20 \\
&= 0.
\end{aligned}$$

We've used that $L[y_1] = 0$ and $L[y_2] = 0$, since y_1 and y_2 are solutions. Thus $y_1 = c_1y_1 + c_2y_2$ is also a solution, which is what we had to prove. ■

We can use this theorem to get new solutions from known ones: if y_1 and y_2 are solutions, then so are $y_3 = (y_1 + y_2)/2$ and $y_2 = (y_1 - y_2)/2$. These are easily obtained by choosing $c_1 = c_2 = 1/2$, and $c_1 = 1/2$, $c_2 = -1/2$ in the theorem.

In order for the theorem to hold, y_1 and y_2 have to be "independent". What does this mean? We'll define this properly soon, but for the moment it suffices to say that y_1 and y_2 are not a multiple of each other. If this happens, say $y_2 = \alpha y_1$, for some constant α, then

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 y_1 + c_2 \alpha y_1 \\ &= (c_1 + c_2\alpha) y_1 \\ &= c_3 y_1, \end{aligned}$$

where $c_3 = c_1 + c_2\alpha$ is another constant. We see that in this case, our proposed general solution y only depends on one constant. That's not enough!

Here's why this theorem absolutely rocks: in order to find the general solution of

$$L[y] = 0,$$

it suffices to find two solutions y_1 and y_2! Awesome!

It's easy to find two such solutions: guess

$$y = e^{\lambda x},$$

for some constant λ, to be determined. Then

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}.$$

Plugging all this in, we get

$$\begin{aligned} a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + c e^{\lambda x} &\stackrel{?}{=} 0 \\ e^{\lambda x}\left(a\lambda^2 + b\lambda + c\right) &\stackrel{?}{=} 0 \\ a\lambda^2 + b\lambda + c &\stackrel{?}{=} 0, \end{aligned}$$

since $e^{\lambda x}$ is never zero. Thus, in order to find solutions, we have to choose λ to be a solution of the quadratic equation

$$\boxed{a\lambda^2 + b\lambda + c = 0.}$$

This equation is known as the **Characteristic equation** of the differential equation. From it, we get two solutions for λ:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This gives two solutions of the original differential equation, namely

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}.$$

Using our theorem, we find that the general solution is

$$\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.}$$

Thus, we've constructed the general solution for a second-order linear equations with constant coefficients, and all we've had to do was solve a quadratic equation!

This works very well if λ_1 and λ_2 are both real, and different. In the other cases, we'll have to do a bit of extra work. Let's look at some examples where the above does work.

Example: Consider the initial-value problem

$$\begin{cases} y'' - y = 0 \\ y(0) = 2, \quad y'(0) = -1 \end{cases},$$

Note that we're specifying two initial conditions, since we have two constants to determine. Let's start with the characteristic equation: we have $a = 1$, $b = 0$, $c = -1$.

$$\lambda^2 - 1 = 0 \quad \Rightarrow \lambda_1 = 1, \;\lambda_2 = -1,$$

from which $y_1 = e^x$, $y_2 = e^{-x}$, and the general solution is

$$y = c_1 e^x + c_2 e^{-x}.$$

Since we'll need y' to use the second initial condition, let's calculate it now: $y' = c_1 e^x - c_2 e^{-x}$. Plugging in the two initial conditions, we get

$$y(0) = c_1 + c_2, \quad y'(0) = c_1 - c_2,$$

so that $c_1 + c_2 = 2$ and $c_1 - c_2 = -1$. Adding and subtracting these two equations we find that $c_1 = 1/2$ and $c_2 = 3/2$. Finally, the solution of the initial-value problem is

$$y = \frac{1}{2}e^x + \frac{3}{2}e^{-x}.$$

Example: Let

$$y'' + 5y' + 6y = 0.$$

The characteristic equation is

$$\begin{aligned} & & \lambda^2 + 5\lambda + 6 = 0 \\ \Rightarrow & & (\lambda + 2)(\lambda + 3) = 0 \\ \Rightarrow & & \lambda_1 = -2, \quad \lambda_2 = -3, \end{aligned}$$

and thus $y_1 = e^{-2x}$, $y_2 = e^{-3x}$. The general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-2x} + c_2 e^{-3x}.$$

Example: Let's consider $y'' - y = 0$ again. We know already that $y_1 = e^x$ and $y_2 = e^{-x}$. From this it follows that $y_3 = 2y_1 = 2e^x$ is also a solution. Could we use y_1 and y_3 to construct the general solution? Let's try.

$$\begin{aligned} y &= c_1 y_1 + c_3 y_3 \\ &= c_1 e^x + c_3 2e^x \\ &= (c_1 + 2c_3)e^x. \end{aligned}$$

Clearly this solution is not the general solution that we are looking for: for one, it does not contain the known solution $y_2 = e^{-x}$ as a special case. The solutions y_1 and y_3 above are called **linearly dependent**.

Definition. Two functions $f(x)$ and $g(x)$ are called **linearly independent** if the equation

$$c_1 f(x) + c_2 g(x) = 0, \quad \text{for all } x,$$

can only be satisfied by choosing $c_1 = 0$, $c_2 = 0$. Two functions that are not linearly independent are called **linearly dependent**.

Example: $f = e^x$ and $g = 2e^x$ are linearly dependent because

$$-2f(x) + g(x) = 0,$$

so $c_1 = -2$ and $c_2 = 1$. If the only choice was to choose them both zero, the functions would be independent.

Theorem 2 *Two functions f and g are linearly dependent if their Wronskian*

$$W(f,g)(x) = f(x)g'(x) - f'(x)g(x) = 0.$$

Proof: If f and g are linearly dependent, then we can find constants c_1 and c_2, not both zero, so that

$$c_1 f + c_2 g = 0, \quad \text{for all } x.$$

Then also

$$c_1 f' + c_2 g' = 0, \quad \text{for all } x.$$

Now, let's assume that $f \neq 0$, otherwise we'll switch the roles of f and g. Then

$$c_1 = -c_2 \frac{g}{f}$$

$$\Rightarrow \quad -c_2 \frac{g}{f} f' + c_2 g' = 0$$

$$\Rightarrow \quad \frac{c_2}{f} (fg' - f'g) = 0.$$

Note that $c_2 \neq 0$, since otherwise c_1 would also be zero, which would imply the functions are linearly independent. Thus

$$fg' - f'g = 0 \quad \Rightarrow \quad W(f,g)(x) = 0,$$

which is what we had to prove. ■

Example: $W(e^x, 2e^x) = e^x(2e^x) - e^x(2e^x) = 0$, since the two functions are linearly dependent.

Example: $W(e^x, e^{-x}) = e^x(-e^{-x}) - e^x(e^{-x}) = -1 - 1 = -2 \neq 0$, since the functions are linearly independent.

Now we can rephrase our awesome theorem more carefully:

Theorem 3 *If y_1 and y_2 are two solutions of $L[y] = 0$ and their Wronskian $W(y_1, y_2) \neq 0$, then the general solution is $y = c_1 y_1 + c_2 y_2$.*

Lecture 9. The Wronskian and linear independence

In the previous lecture, we learned how to solve

$$ay'' + by' + cy = 0,$$

using three steps:

1. Characteristic equation: $a\lambda^2 + b\lambda + c = 0$
2. Fundamental solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$
3. General solution by superposition: $y = c_1 y_1 + c_2 y_2$

We restricted ourselves mainly to constant-coefficient equations, but our superposition theorem also holds for equations of the form

$$y'' + p(x)y' + q(x)y = 0.$$

Also in this case, the general solution is given by $y = c_1 y_1 + c_2 y_2$, where y_1 and y_2 are two linearly independent solutions of $y'' + p(x)y' + q(x)y = 0$. To see if two solutions are linearly independent, we calculate their Wronskian

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2.$$

If $W(y_1, y_2) \neq 0$, the two solutions y_1 and y_2 are linearly independent.

We've already seen how to solve constant-coefficient equations when the roots of the characteristic equation are real and different. In this lecture, we'll see how to solve the case where the two roots λ_1 and λ_1 are equal: $\lambda_1 = \lambda_2$ (note that they are automatically real in this case). It is clear that our previous plan of attach won't work, since we now have $y_1 = y_2$, and we don't have two linearly independent solutions.

Theorem 4 (Abel's theorem) *Let y_1 and y_2 be any two solutions of $y' + p(x)y' + q(x)y = 0$, then*

$$W(y_1, y_2) = ce^{-\int p(x)dx},$$

where c is a constant.

Proof: By definition $W(y_1, y_2) = y_1 y_2' - y_1' y_2$. Thus

$$\begin{aligned}
W' &= y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' \\
&= y_1 y_2'' - y_1'' y_2 \\
&= y_1(-p(x)y_2' - q(x)y_2) - y_2(-p(x)y_1' - q(x)y_1) \\
&= -p(x)(y_1 y_2' - y_1' y_2) - q(x)y_1 y_2 + q(x)y_2 y_2 \\
&= -p(x)W,
\end{aligned}$$

from which

$$
\begin{aligned}
& & \frac{dW}{dx} &= -p(x)W \\
&\Rightarrow & \int \frac{dW}{W} &= -\int p(x)dx + \alpha \\
&\Rightarrow & \ln W &= -\int p(x)dx + \alpha \\
&\Rightarrow & W &= e^{-\int p(x)dx+\alpha} \\
&\Rightarrow & W &= ce^{-\int p(x)dx},
\end{aligned}
$$

where α and $c = e^{\alpha}$ are constants. This is what we had to prove. ■

Hence, if y_1 and y_2 are two linearly dependent solutions, $c = 0$ in Abel's theorem.

We can now use Abel's theorem to get a second linearly independent solution of a second-order linear differential equation if we already know a first one. This is known as **reduction of order**, because it reduces the problem of finding a solution of a second-order equation to that of solving a related first-order equation. Here's how this works: supposed we know y_1, but we don't know y_2. Then, from Abel's theorem we have that

$$y_1 y_2' - y_1' y_2 = W = ce^{-\int p(x)dx}.$$

Note that the entire right-hand side is known. We have to choose c to be nonzero, since we want a linearly-independent second solution. This equation is nothing but a first-order linear equation for y_2, given y_1. This is the announced reduction of order. Let's solve this first-order equation: dividing by y_1^2 we get

$$
\begin{aligned}
& & \frac{1}{y_1} y_2' - \frac{y_1'}{y_1^2} y_2 &= \frac{W}{y_1^2} \\
&\Rightarrow & \frac{d}{dx}\left(\frac{y_2}{y_1}\right) &= \frac{W}{y_1^2} \\
&\Rightarrow & \frac{y_2}{y_1} &= \int \frac{W}{y_1^2} dx \\
&\Rightarrow & y_2 &= y_1 \int \frac{W}{y_1^2} dx.
\end{aligned}
$$

We don't really care about the integration constant in this integral, as we're only interested in finding one extra solution. With this second solution, we may construct the general solution using the superposition theorem. Thus, to find this second solution, we first use Abel's theorem to find the Wronskian W, after which we use

$$\boxed{y_2 = y_1 \int \frac{W}{y_1^2} dx}$$

to get the second solution.

Example: Consider the equation $2x^2 y'' + 3xy' - y = 0$, for $x > 0$. Let's check that $y_1 = 1/x$ is a solution of this equation:

$$y_1' = -\frac{1}{x^2}, \quad y_1'' = \frac{2}{x^3},$$

from which

$$2x^2 y_1'' + 3xy_1' - y_1 = 2x^2 \frac{2}{x^3} - 3x\frac{1}{x^2} - \frac{1}{x} = \frac{4}{x} - \frac{4}{x} - \frac{1}{x} \stackrel{!}{=} 0.$$

Thus $y_1 = 1/x$ is a solution. Now we use the method above to find a second solution y_2, linearly independent of the first one. First we need to compute the Wronskian W. Using Abel's theorem

$$W = ce^{-\int p(x)dx}.$$

For our equation $p(x) = 3x/(2x^2) = 3/(2x)$, since we need to write the differential equation so that the coefficient of y'' is one, in order to use Abel's theorem. Thus

$$W = ce^{-\frac{3}{2}\int \frac{1}{x}dx} = ce^{-\frac{3}{2}\ln x} = cx^{-3/2}.$$

There's no need to choose c at this point. We can choose it later when it is convenient. The reduction-of-order formula gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{W}{y_1^2} dx \\
&= \frac{1}{x} \int \frac{cx^{-3/2}}{x^{-2}} dx \\
&= \frac{c}{x} \int x^{1/2} dx \\
&= \frac{c}{x} \frac{x^{3/2}}{3/2} \\
&= \frac{2}{3} cx^{1/2} \\
&= \sqrt{x},
\end{aligned}$$

where we have chosen $c = 3/2$, for convenience. Any choice but zero will work. In summary, our general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 \frac{1}{x} + c_2 \sqrt{x}.$$

Example: This is an **important example**: it will allow us to solve the case of linear, constant-coefficient equations where the roots of the characteristic equation are equal: $\lambda_1 = \lambda_2$. Thus, let's start with such an equation:

$$ay'' + by' + cy = 0,$$

with characteristic equation $a\lambda^2 + b\lambda + c = 0$, and

$$\lambda_1 = \lambda_2 = -\frac{b}{2a}, \quad \text{and} \quad b^2 = 4ac.$$

Our original method only results in one solution

$$y_1 = e^{\lambda_1 x}.$$

Let's use the reduction-of-order method to get a second one. First, we use Abel's theorem to calculate the Wronskian. Note that we have $p(x) = b/a$. Also, let's use α for the constant in Abel's theorem, as the equation already has a c in it.

$$\begin{aligned} W &= \alpha e^{-\int p(x)dx} \\ &= \alpha e^{-bx/a}. \end{aligned}$$

Using reduction of order,

$$\begin{aligned} y_2 &= y_1 \int \frac{W}{y_1^2} dx \\ &= e^{\lambda_1 x} \int \frac{\alpha e^{-bx/a}}{e^{2\lambda_1 x}} dx \\ &= \alpha e^{\lambda_1 x} \int e^{-(\frac{b}{a}+2\lambda_1)x} dx \\ &= \alpha e^{\lambda_1 x} \int dx \\ &= \alpha e^{\lambda_1 x} x \\ &= x e^{\lambda_1 x}. \end{aligned}$$

We have chosen $\alpha = 1$, and used that $\lambda_1 = -b/(2a)$. Thus, we get a second, linearly independent solution by multiplying the first one by x. That's easy enough. The general solution of a linear, second-order equation with constant coefficients that has both roots of the characteristic equation equal is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} = (c_1 + c_2 x) e^{\lambda_1 x}.$$

Example: Consider the equation $y'' + 2y' + y = 0$. Its characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$, from which it follows that $\lambda_1 = \lambda_2 = -1$. Thus $y_1 = e^{-x}$, and $y_2 = xe^{-x}$, using the result from the previous example. The general solution is

$$y = c_1 y_1 + c_2 y_2 = (c_1 + c_2 x) e^{-x}.$$

Lecture 10. Complex roots of the characteristic equation

In this lecture, we'll see how to solve

$$ay'' + by' + cy = 0,$$

when the characteristic equation $a\lambda^2 + b\lambda + c = 0$ has complex roots. Then

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

with $b^2 - 4ac < 0$, or $4ac - b^2 > 0$. Thus

$$\lambda_{1,2} = \frac{-b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a},$$

where i is the square root of unity, $i^2 = -1$. It is also referred to as the imaginary unit. Following what we did before, we have two solutions

$$\begin{array}{lll} & y_1 = e^{\lambda_1 x}, & y_2 = e^{\lambda_2 x} \\ \Rightarrow & y_1 = e^{(\alpha+i\beta)x}, & y_2 = e^{(\alpha-i\beta)x}, \end{array}$$

with $\alpha = -b/(2a)$, the real part of λ_1 or λ_2, and $\beta = \sqrt{4ac - b^2}/(2a)$, the imaginary part of λ_1. Thus

$$y_1 = e^{\alpha x}e^{i\beta x}, \quad y_2 = e^{\alpha x}e^{-i\beta x}.$$

But what does $e^{\pm i\beta x}$ mean? We'll now see two different ways to convince you that

$$\boxed{\begin{array}{rcl} e^{i\beta x} & = & \cos\beta x + i\sin\beta x, \\ e^{-i\beta x} & = & \cos\beta x - i\sin\beta x. \end{array}}$$

These are known as **Euler's formulae**.

Method 1: power series

Using the Taylor series of the exponential and the trig functions, we get

$$\begin{aligned} e^{i\beta x} &= 1 + i\beta x + \frac{(i\beta x)^2}{2!} + \frac{(i\beta x)^3}{3!} + + \frac{(i\beta x)^4}{4!} + \cdots \\ &= 1 + i\beta x - \frac{\beta^2 x^2}{2!} - i\frac{\beta^3 x^3}{3!} + \frac{\beta^4 x^4}{4!} + \cdots \\ &= 1 - \frac{\beta^2 x^2}{2!} + \frac{\beta^4 x^4}{4!} + \cdots + i\left(\beta x - \frac{\beta^3 x^3}{3!} + \cdots\right) \\ &= \cos\beta x + i\sin\beta x. \end{aligned}$$

This proves the first formula. However, throughout, we have assumed that the Taylor series that we know to be valid for real arguments are also valid for complex arguments. So the above is not exactly waterproof.

Using the symmetry properties of the trig functions, we have

$$e^{-i\beta x} = \cos(-\beta x) + i\sin(-\beta x) = \cos\beta x - i\sin\beta x,$$

which proves the second formula.

Method 2: differential equations

Let $u = e^{i\beta x}$, then

$$u' = i\beta e^{i\beta x}, \quad u'' = (i\beta)^2 e^{i\beta x} = -\beta^2 e^{i\beta x}.$$

We've assumed that the derivative rules for the exponential hold as before, even though the exponent is not real. There's some logic to this: if we intend to extend the definition of the exponential to complex arguments, then we should insist that the properties we hold to be true, remain true in this more general setting. Otherwise we're not gaining much by doing this extension. Proceeding, we see that $u = e^{i\beta x}$ satisfies the differential equation

$$u'' + \beta^2 u = 0,$$

with initial conditions $u(0) = 1$, $u'(0) = i\beta$. These were obtained by plugging in $x = 0$ to our explicit expressions for u and u'. Thus, $u(x)$ is the unique solution of the initial-value problem

$$\begin{cases} u'' + \beta^2 u = 0 \\ u(0) = 1, \quad u'(0) = i\beta. \end{cases}$$

Now consider $v = \cos\beta x + i\sin\beta x$. Then

$$\begin{aligned} v' &= -\beta\sin\beta x + i\beta\cos\beta x, \\ v'' &= -\beta^2\cos\beta x - i\beta^2\sin\beta x = -\beta^2 v. \end{aligned}$$

Further, by plugging in $x = 0$ we get $v(0) = 1$, $v'(0) = i\beta$. In other words, $v(x)$ satisfies the same initial-value problem that $u(x)$ satisfies. We have to conclude that $u(x) = v(x)$, which is what we wanted to prove.

Thus our two fundamental solution are

$$\begin{cases} y_1 &= e^{\alpha x}(\cos\beta x + i\sin\beta x), \\ y_2 &= e^{\alpha x}(\cos\beta x - i\sin\beta x). \end{cases}$$

Unfortunately these are not real valued. Awesome theorem to the rescue! Indeed, if y_1 and y_2 are solutions, then so are

$$y_3 = \frac{y_1 + y_2}{2} \quad \text{and} \quad y_4 = \frac{y_1 - y_2}{2i}.$$

This gives

$$y_3 = e^{\alpha x}\cos\beta x, \quad y_4 = e^{\alpha x}\sin\beta x.$$

Using these as fundamental solutions, the general solution is

$$\begin{aligned} y &= c_1 y_3 + c_2 y_4 \\ &= c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x). \end{aligned}$$

Example: Consider the equation $y'' + y' + y = 0$. Its characteristic equation is

$$\begin{aligned} \lambda^2 + \lambda + 1 &= 0 \\ \Rightarrow \qquad \lambda_{1,2} &= \frac{-1 \pm \sqrt{-3}}{2} \\ &= \frac{-1 \pm i\sqrt{3}}{2} \\ \Rightarrow \qquad \alpha = -\frac{1}{2}, \quad \beta &= \frac{\sqrt{3}}{2}, \end{aligned}$$

so that the fundamental solutions are

$$y_1 = e^{-x/2} \cos \frac{\sqrt{3}}{2} x, \quad y_2 = e^{-x/2} \sin \frac{\sqrt{3}}{2} x.$$

The general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{-x/2} \cos \frac{\sqrt{3}}{2} x + c_2 e^{-x/2} \sin \frac{\sqrt{3}}{2} x. \end{aligned}$$

Summary of solving constant coefficient equations

To solve the linear second-order, constant-coefficient equation

$$ay'' + by' + cy = 0$$

we proceed using the following steps:

1. Solve the characteristic equation:

$$a\lambda^2 + b\lambda + c = 0.$$

   This equation has two roots, λ_1 and λ_2.

2. Write down the fundamental solutions:

   i) $\lambda_1 \neq \lambda_2$, and both are real. Then

$$y_1 = e^{\lambda_1 x}, \qquad y_2 = e^{\lambda_2 x}.$$

   ii) $\lambda_1 = \lambda_2$ (then they're both real). Then

$$y_1 = e^{\lambda_1 x}, \quad y_2 = xe^{\lambda_1 x}.$$

   iii) $\lambda_1 \neq \lambda_2$, complex conjugates of each other: $\lambda_{1,2} = \alpha \pm i\beta$. Then

$$y_1 = e^{\alpha x} \cos \beta x, \qquad y_2 = e^{\alpha x} \sin \beta x.$$

3. Write down the general solution:

$$y = c_1 y_1 + c_2 y_2.$$

Lecture 11. Euler equations

So far, we've mainly looked at equations with constant coefficients:

$$ay'' + by' + cy = 0.$$

Many of our results are valid for more general linear equations of the form

$$y'' + p(x)y' + q(x)y = 0.$$

An important class of equations where we can also solve everything explicitly is the class of **Euler equations**. These are of the form

$$x^2y'' + \alpha xy' + \beta y = 0, \qquad x > 0.$$

So here $p = \alpha x/x^2 = \alpha/x$ and $q = \beta/x^2$.

In this case, we guess fundamental solutions of the form

$$y = x^s.$$

Then

$$\begin{aligned} && y' &= sx^s \\ \Rightarrow && y'' &= s(s-1)x^{s-2}. \end{aligned}$$

Substituting this into our differential equations, we get

$$\begin{aligned} && x^2y'' + \alpha xy' + \beta y &= 0 \\ \Rightarrow && x^2s(s-1)x^{s-2} + \alpha xsx^{s-1} + \beta x^s &= 0 \\ \Rightarrow && [s(s-1) + \alpha s + \beta]\, x^s &= 0, \end{aligned}$$

and thus we need to impose that s is chosen so that

$$\boxed{s^2 + (\alpha - 1)s + \beta = 0}.$$

This is called the **indicial equation**. As in the case of equations with constant coefficients there are three cases, depending on the solutions of this quadratic equation. The roots are

$$s_{1,2} = \frac{1 - \alpha \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}.$$

Case 1: $s_1 \neq s_2$, both real

Then $y_1 = x^{s_1}$ and $y_2 = x^{s_2}$. The general solution is

$$y = c_1x^{s_1} + c_2x^{s_2}.$$

Case 2: $s_1 = s_2$, real

Then $y_1 = x^{s_1}$. In order to get a a second linearly independent solution, we need to use reduction of order. In order to do so, we use Abel's theorem first. We get

$$\begin{aligned} W &= ce^{-\int p(x)dx} \\ &= ce^{-\int \frac{\alpha}{x}dx} \\ &= ce^{-\alpha \ln x} \\ &= ce^{\ln x^{-\alpha}} \\ &= cx^{-\alpha}. \end{aligned}$$

Using the reduction-of-order formula, we get

$$\begin{aligned} y_2 &= y_2 \int \frac{W}{y_1^2} dx \\ &= x^{s_1} \int \frac{cx^{-\alpha}}{x^{2s_1}} dx \\ &= cx^{s_1} \int x^{-\alpha-2s_1} dx. \end{aligned}$$

But $s_1 = (1-\alpha)/2$, which means that $2s_1 = 1 - \alpha$, so the exponent of the integrand is -1. Thus

$$y_2 = cx^{s_1} \int x^{-1} dx = cx^{s_1} \ln x = x^{s_1} \ln x,$$

where we have chosen $c = 1$. The general solution in this case is

$$y = c_1 x^{s_1} + c_2 x^{s_1} \ln x.$$

Case 3: $s_{1,2}$ complex

Then $s_{1,2} = \eta \pm i\mu$. Then x^{s_1} is a solution, as is x^{s_2}. Unfortunately, these are complex solutions. We'd prefer to have real-valued solutions. We've seen before that for a linear equation, the real part of a complex-valued solution, as well as its imaginary part are both solutions in their own right. This is because these solutions are almost (up to constant factors of 2 and $2i$) sums and differences of the two complex solutions.

Let's see what these real and imaginary parts are in this case. We have

$$\begin{aligned} x^{s_1} &= x^{\eta+i\mu} \\ &= x^{\eta} x^{i\mu} \\ &= x^{\eta} e^{\ln x^{i\mu}} \\ &= x^{\eta} e^{i\mu \ln x} \\ &= x^{\eta} \left(\cos(\mu \ln x) + i \sin(\mu \ln x) \right), \end{aligned}$$

so that

$$\left\{ \begin{array}{lcl} y_1 &=& x^{\eta} \cos(\mu \ln x) \\ y_2 &=& x^{\eta} \sin(\mu \ln x) \end{array} \right.$$

are two linearly independent fundamental solutions. The general solution is

$$y = c_1 x^\eta \cos(\mu \ln x) + c_2 x^\eta \sin(\mu \ln x).$$

This covers all cases for the Euler equation.

Example: Consider the equation $x^2 y'' + y = 0$. This is an Euler equation with $\alpha = 1$, $\beta = 0$. The indicial equation is

$$s^2 - s + 1 = 0 \quad \Rightarrow \quad s_{1,2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

The fundamental solutions are

$$\begin{cases} y_1 &= x^{1/2} \cos \dfrac{\sqrt{3}}{2} x = \sqrt{x} \cos \dfrac{\sqrt{3}}{2} x, \\ \\ y_2 &= x^{1/2} \sin \dfrac{\sqrt{3}}{2} x = \sqrt{x} \sin \dfrac{\sqrt{3}}{2} x, \end{cases}$$

and the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 \sqrt{x} \cos \frac{\sqrt{3}}{2} x + c_2 \sqrt{x} \sin \frac{\sqrt{3}}{2} x.$$

Example: Consider the equation $x^2 y'' - 3xy' + 4y = 0$. Here $\alpha = -3$, $\beta = 4$. The indicial equation is

$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_1 = s_2 = 2.$$

As a consequence, our fundamental solutions are

$$y_1 = x^2, \quad y_2 = x^2 \ln x,$$

and the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 x^2 + c_2 x^2 \ln x.$$

Lecture 12. Nonhomogeneous equations: undetermined coefficients

Now that we know how to solve the homogeneous equation

$$L[y] = ay'' + by' + cy = 0,$$

let's investigate how we could go about solving the nonhomogeneous equation

$$L[y] = ay'' + by' + cy = g(x).$$

Theorem 5 *The general solution of $L[y] = g(x)$ is $y = y_H + y_P$, where $y_H = c_1y_1 + c_2y_2$ is the general solution of the homogeneous problem $L[y] = 0$, and y_P is any particular solution of $L[y] = g(x)$.*

Note that y_1 and y_2 are the fundamental solutions of the homogeneous problems we've been discussing up to this point. What this theorem says is that in order to solve the nonhomogeneous problem, we have to solve the homogeneous problem first. Having done so, all we need is **one** solution of the full equation. Let's prove this theorem.

Proof:

$$L[y] = L[y_H + y_P] = L[y_H] + L[y_P] = 0 + L[y_P] = g(x).$$

Furthermore, y depends on two constants c_1 and c_2, which is required for a second-order problem. This finishes the proof. ■

What would be the effect of you and your neighbor picking different particular solutions y_{P_1} and y_{P_2}? The answer is that it doesn't matter. Here's why: calculate

$$L[y_{P_1} - y_{P_2}] = L[y_{P_1}] - L[y_{P_2}] = g(x) - g(x) = 0.$$

We have to conclude that the difference of any two particular solutions $y_{P_1} - y_{P_2}$ is a solution of the homogeneous problem. But all solutions of the homogeneous problems can be written as a linear combination of the fundamental solutions y_1 and y_2. Thus

$$y_{P_1} - y_{P_2} = c_3y_1 + c_4y_2.$$

Constructing the general solution with y_{P_1} or y_{P_2} gives

$$\begin{aligned}
y_{P_1}: \qquad & y = c_1y_1 + c_2y_2 + y_{P_1} \\
& = c_1y_1 + c_2y_2 + y_{P_2} + c_3y_1 + c_4y_2 \\
& = (c_1 + c_3)y_1 + (c_2 + c_4)y_2 + y_{P_2} \\
y_{P_2}: \qquad & y = c_1y_1 + c_2y_2 + y_{P_2}.
\end{aligned}$$

Thus, working with a different particular solution only affects the value of the arbitrary constants. So, it is irrelevant which particular solution we use.

Example: Consider the equation $y''+3y'-4y = 1$. We need to calculate the homogeneous solution first. The characteristic equation is

$$\lambda^2 + 3\lambda - 4 = 0 \quad \Rightarrow \quad (\lambda + 4)(\lambda - 1) = 0 \quad \Rightarrow \quad \lambda_1 = -4,\ \lambda_2 = 1.$$

Thus the homogeneous solution is

$$y_H = c_1 e^{-4x} + c_2 e^x.$$

Next we need to find a particular solution. Since the right-hand side is just a constant, maybe guessing a constant will work? Let's give it a shot: guess

$$\begin{aligned} && y_P &= A \\ \Rightarrow && y_P' &= 0 \\ \Rightarrow && y_P'' &= 0. \end{aligned}$$

Substituting these into the differential equation gives

$$-4A = 1 \quad \Rightarrow \quad A = -\frac{1}{4}.$$

Thus $y_P = -1/4$, and the general solution is

$$y = y_H + y_P = c_1 e^{-4x} + c_2 e^x - \frac{1}{4}.$$

That was not too hard. Now, **how do we find particular solutions**? Will it always be this easy? No. We'll see a general method for finding particular solutions in the next lecture. However, in many cases there is a better (meaning simpler) method. The previous example illustrates the principle of this method, which is known as the method of **undetermined coefficients**: based on the form of the right-hand side, we choose a form for a particular solution. This form will have some undetermined coefficients in it. Plugging in gives us conditions on these coefficients so that our guess actually works. It may happen that it is not possible to satisfy the conditions we get and our guess fails. In order to prevent this from happening, we'd like to come up with some rules that tell us how to guess. We've already seen one such rule:

$$\boxed{g(x) \text{ is constant} \Rightarrow \text{guess a constant: } y_P = A.}$$

Example: Consider the equation $y'' + 3y' - 4y = 3e^{2x}$. This equation has the same homogeneous part as the previous one, thus $y_H = c_1 e^{-4x} + c_2 e^x$. Since the derivative of an exponential returns the same exponential, we could try to guess an exponential for the particular solution. It should cancel from the entire equation, leaving us with a condition on its coefficient. So, let's try

$$\begin{aligned} && y_P &= Ae^{2x} \\ \Rightarrow && y_P' &= 2Ae^{2x} \\ \Rightarrow && y_P'' &= 4Ae^{2x}. \end{aligned}$$

Substitution in the equation gives

$$\begin{aligned} && \left(4Ae^{2x}\right) + 3\left(2Ae^{2x}\right) - 4\left(Ae^{2x}\right) &= 3e^{2x} \\ \Rightarrow && 4A + 6A - 4A &= 3 \\ \Rightarrow && 6A &= 3 \\ \Rightarrow && A &= \frac{1}{2}, \end{aligned}$$

and our guess for the particular solution works if we choose $A = 1/2$. Thus

$$y_P = \frac{1}{2}e^{2x},$$

and the general solution is

$$u = y_H + y_P = c_1 e^{-4x} + c_2 e^x + \frac{1}{2}e^{2x}.$$

This leads us to the **exponential rule**:

$$\boxed{g(x) = \alpha e^{\kappa x} \Rightarrow \text{guess the same exponential: } y_P = Ae^{\kappa x}.}$$

A third rule is the **Polynomial rule**:

$$\boxed{\begin{array}{c} g(x) = \alpha_N x^N + \alpha_{N-1}x^{N-1} + \cdots \alpha_1 x + \alpha_0 \quad \Rightarrow \\ y_P = A_N x^N + A_{N-1}x^{N-1} + \cdots A_1 x + A_0. \end{array}}$$

This rule makes sense: in order to get an x^N on the right, we definitely need to put that degree in on the left. However, upon doing so, we'll get terms of lower degree on the left, because of the derivatives. That's why we need all terms of the polynomial, even though they may not appear on the right-hand side.

Example: Consider $y'' + 3y' - 4y = 4x^2$. Again $y_H = c_1 e^{-4x} + c_2 e^x$. The nonhomogeneous part is a polynomial of degree 2, so we'll guess

$$\begin{aligned} & & y_P &= Ax^2 + Bx + C \\ &\Rightarrow & y_P' &= 2Ax + B \\ &\Rightarrow & y_P'' &= 2A. \end{aligned}$$

Substituting this in the equation gives

$$2A + 3(2Ax + B) - 4(Ax^2 + Bx + C) = 4x^2 - 1.$$

This equation has to hold for all x. In order to satisfy it, we can equate the coefficients of equal powers of x on both sides. We get:

$$\begin{aligned} x^2 : & \quad & -4A &= 4 \\ x^1 : & \quad & 6A - 4B &= 0 \\ x^0 : & \quad & 2A + 3B - 4C &= 0. \end{aligned}$$

This gives $A = -1$, $B = -3/2$, and $C = -13/8$. We find

$$y_P = -x^2 - \frac{3}{2}x - \frac{11}{8}.$$

We see that determining these coefficients merely requires some simple algebra. It may have been tempting to just try $y_P = Ax^2$, since only an x^2 term appears in the right-hand side. Note that this would not have worked. If you're not convinced, try it!

A fourth rule is the **Exponential-polynomial rule**:

$$\boxed{\begin{array}{c} g(x) = e^{\beta x}\left(\alpha_N x^N + \alpha_{N-1}x^{N-1} + \cdots \alpha_1 x + \alpha_0 \ \right) \Rightarrow \\ y_P = e^{\beta x}\left(A_N x^N + A_{N-1}x^{N-1} + \cdots A_1 x + A_0\right). \end{array}}$$

This rule is also sensible: every time we'll take derivative of an exponential multiplied by an polynomial of degree N, we'll get back the same exponential, multiplied by a different polynomial of degree N. Even if $g(x)$ only contains a few terms of the polynomial, we'll include all terms of lower degree in our guess for y_P as well, as before.

A fifth rule is the **Cosine-sine rule**:

$$\boxed{\begin{array}{c} g(x) = \alpha\cos\omega x + \beta\sin\omega x \Rightarrow \\ y_P = A\cos\omega x + B\sin\omega x. \end{array}}$$

Every time we'll take a derivative of a sine or cosine, we'll get the other one. This is why we include both, even when $g(x)$ contains only one of them.

Example: Consider $y'' + 3y' - 4y = 2\sin x$. Using the Cosine-sine rule, we guess

$$\begin{aligned} & & y_P &= A\cos x + B\sin x \\ &\Rightarrow & y_P' &= -A\sin x + B\cos x \\ &\Rightarrow & y_P'' &= -A\cos x - B\sin x. \end{aligned}$$

Substituting these in our equation gives

$$-A\cos x - B\sin x + 3\left(-A\sin x + B\cos x\right) - 4\left(A\cos x + B\sin x\right) = 2\sin x.$$

Equating the coefficients of $\sin x$ and $\cos x$ gives two equations for A and B:

$$\begin{array}{llcl} \sin x : & -B - 3A - 4B = 2 & \Rightarrow & -3A - 5B = 2, \\ \cos x : & -A + 3B - 4A = 0 & \Rightarrow & -5A + 3B = 0. \end{array}$$

Solving these equations gives $A = -3/17$ and $B = -5/17$ (check!) so that the particular solution is

$$y_P = -\frac{3}{17}\cos x - \frac{5}{17}\sin x.$$

Note that we needed both $\sin x$ and $\cos x$ in order to find a particular solution, although only $\sin x$ appeared in $g(x)$.

A sixth rule is the **Polynomial-Cosine-sine rule**:

$$\boxed{\begin{array}{c} g(x) = P_n(x)\cos\omega x + Q_m(x)\sin\omega x \Rightarrow \\ y_P = S_N(x)\cos\omega x + T_N(x)\sin\omega x. \end{array}}$$

Here $P_n(x)$ and $Q_m(x)$ are given polynomials of degree n and m respectively. Then $S_N(x)$ and $T_N(x)$ are polynomials of degree N, where N is the maximum of n and m. Seeing that this rule works is an easy consequence of the product rule and the previous guessing rules.

Finally we have the seventh rule, the mother of all rules: the **Polynomial-sine-cosine-exponential rule**.

$$\boxed{\begin{array}{c} g(x) = e^{\alpha x}\left(P_n(x)\cos\omega x + Q_m(x)\sin\omega x\right) \Rightarrow \\ y_P = e^{\alpha x}\left(S_N(x)\cos\omega x + T_N(x)\sin\omega x\right). \end{array}}$$

As for the previous rule, $P_n(x)$ and $Q_m(x)$ are given polynomials of degree n and m respectively. Then $S_N(x)$ and $T_N(x)$ are polynomials of degree N, where N is the maximum of n and m. This rule encompasses all the previous rules as special cases.

All of these rules are very sensible, once we realize that taking derivative of our guesses results in expressions that have similar terms. However: **this does not always work!** What could go wrong?

Example: Consider the differential equation $y'' + 3y' - 4y = e^{-4x}$. Based on the exponential rule, we'd guess

$$\begin{aligned} & & y_P &= Ae^{-4x} \\ &\Rightarrow & y_P' &= -4Ae^{-4x} \\ &\Rightarrow & y_P'' &= 16Ae^{-4x}. \end{aligned}$$

Substitution results in

$$16Ae^{-4x} - 12Ae^{-4x} - 4Ae^{-4x} = e^{-4x} \quad \Rightarrow \quad 0 = e^{-4x}!$$

Not a satisfactory conclusion, to say the least! What went wrong? Remember that our homogeneous solution is $y_H = c_1e^x + c_2e^{-4x}$. This says that any multiple of e^{-4x} is a solution of the homogeneous equation. Thus, if we're going to plug in $y_P = Ae^{-4x}$, we know everything will cancel, and there's no way we'll get an equation that will determine A. How do we fix this?

It turns out there's an easy fix: whenever any term occuring in our guess for the particular solution also appears in the homogeneous solution, we multiply our **entire** guess by x.

Example: Let's see how this works for our previous example. Multiplying our entire guess by x gives

$$\begin{aligned} & & y_P &= Axe^{-4x} \\ &\Rightarrow & y_P' &= -4Axe^{-4x} + Ae^{-4x} \\ &\Rightarrow & y_P'' &= 16Ae^{-4x} - 8Ae^{-4x}. \end{aligned}$$

Now we substitute these (somewhat more complicated) expressions into our differential equation. This gives

$$\begin{aligned} & & 16Ae^{-4x} - 8Ae^{-4x} - 12Axe^{-4x} + 3Ae^{-4x} - 4Axe^{-4x} &= e^{-4x} \\ &\Rightarrow & -8Ae^{-4x} + 3Ae^{-4x} &= e^{-4x} \\ &\Rightarrow & -5A &= 1 \\ &\Rightarrow & A &= -\frac{1}{5}, \end{aligned}$$

so that the particular solution is

$$y_P = -\frac{1}{5}xe^{-4x},$$

and the general solution is

$$y = c_1e^x + c_2e^{-4x} - \frac{1}{5}xe^{-4x}.$$

So, in general, we use our seven rules to give an initial guess for the form of y_P. Next, if any term in this guess appears in the homogeneous solution, we multiply the entire guess by x. Now, we check if any of the new terms still appear in the homogeneous solution. If so, we multiply by x again, and so on.

Example: Consider the equation $y''+y = \cos x$. You'll easily check the the homogeneous solution is

$$y_H = c_1 \cos x + c_2 \sin x.$$

Based on the form of $g(x)$, we'd guess $y_P = A\cos x + B\sin x$. But both terms are in the homogeneous solution, thus we modify our guess to

$$y_P = x\left(A\cos x + B\sin x\right),$$

which you should check will work.

Example: Consider $y'' - 2y' + y = e^x(x^2+1)$. You'll find that the homogeneous solution is

$$y_H = c_1e^x + c_2xe^x.$$

Based on the form of $g(x)$, we'll guess $y_P = (Ax^2 + Bx + C)e^x$. But the second (Bxe^x) and third (Ce^x) terms appear in the homogeneous solution, thus we multiply our guess by x: $y_P = x(Ax^2 + Bx + C)e^x = (Ax^3 + Bx^2 + Cx)e^x$. But now the new third term (Cxe^x) still appears in y_H, thus we multiply by x again, so that

$$y_P = x(Ax^3 + Bx^2 + Cx)e^x = (Ax^4 + Bx^3 + Cx^2)e^x,$$

which works (check!).

If $g(x) = g_1(x) + g_2(x)$, where we have rules that tell us what particular solution to guess for $g_1(x)$ and $g_2(x)$, we can simply separate $g(x)$ in these two parts, solve the two simpler problems, and add up the two particular solutions:

Theorem 6 *If $g(x) = g_1(x) + g_2(x)$, then a particular solution for $L[y] = g(x)$ is given by $y_P = y_{P1} + y_{P2}$, where y_{P1} and y_{P2} are particular solutions corresponding to $g_1(x)$ and $g_2(x)$, respectivelly.*

Proof:

$$L[y_P] = L[y_{P1} + y_{P2}] = L[y_{P1}] + L[y_{P2}] = g_1(x) + g_2(x) = g(x),$$

which is what we had to prove. ■

In the next lecture, we'll discuss a method that works for all possible $g(x)$'s. However, when the methods of this lecture work, they're simpler than the method we'll see in the next lecture.

Lecture 13. Nonhomogeneous equations: variation of parameters

In the last lecture, we saw how to solve the equation

$$ay'' + by' + cy = g(x),$$

in the case when $g(x)$ is a polynomial combination of exponentials, sines, cosines, and polynomials. Although these cases are very important for applications, we'd still like to know what to do when $g(x)$ is not of this form. In this lecture, we'll see a general method to answer this question. We'll do even better: the method we'll use gives us a particular solution to any equation of the form

$$y'' + p(x)y' + q(x)y = g(x),$$

even when $p(x)$ and $q(x)$ are not constant. To do this, we'll need two linearly independent solutions y_1 and y_2 of the homogeneous problem

$$y'' + p(x)y' + q(x)y = 0.$$

We know that to find the general solution, all we need is the solution of this homogeneous problem, and one particular solution y_p. Then the general solution is given by

$$y = c_1y_1 + c_2y_2 + y_P.$$

Since we've already seen how to get y_2 if we know y_1 (using Abel's formula, reduction of order), this in effect means that after today's lecture we'll be able to solve any linear second-order equation, as long as we manage to somehow find one solution of the homogeneous problem.

Variation of parameters

This is a general method to construct one y_P, given y_1 and y_2. It works for any $g(x)$. You should realize that when the method of undetermined coefficients can be used, it is a lot easier.

The general solution to the homogeneous problem is

$$y = c_1y_1 + c_2y_2,$$

where c_1 and c_2 are constants. We know that if we plug this in to the whole equation (with $g(x)$), then the left-hand side vanishes. That's not quite what we want, but it's not altogether terrible: we do want a lot of things to cancel. What would happen if c_1 and c_2 were not constant, but were allowed to depend on x? This is the premice of the method of **variation of parameters**: maybe we can fabricate a new solution out of an old solution to a related problem, by letting whatever parameters are present in that solution vary. Here those parameters are c_1 and c_2. So, let's try to construct a particular solution

$$y_P = u_1(x)y_1(x) + u_2(x)y_2(x).$$

So, we're assuming that y_P has a form similar to the homogeneous solution, but a little more complicated. One thing to note before we proceed: we're trying to find one particular solution, and we've introduced two new unknown functions u_1 and u_2. This means that we still get to introduce one constraint between these two functions. We'll do this when it'll be convenient, see below.

Let's substitute our form of the particular solution into the equation. To that end, we need y_P' and y_P''. So, let's take a derivative:

$$y_P' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2.$$

Our original equation for y_P' is of second order. If we'll take another derivative of y_P', we'll see that y_P'' depends on u_1'' and u_2''. Thus, the differential equation we'll have to solve for u_1 and u_2 will also be of second order. That's not any simpler than the problem we started with. This is where we'll impose another condition on u_1 and u_2. We impose that

$$u_1' y_1 + u_2' y_2 = 0.$$

Then

$$y_P' = u_1 y_1' + u_2 y_2',$$

and y_P'' will not depend on second derivatives of u_1 or u_2:

$$y_P'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'.$$

Now we plug everything into our differential equation:

$$\begin{aligned} & & y_P'' + p y_P' + q y_P = g \\ \Rightarrow & & u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2' + p\,(u_1 y_1' + u_2 y_2') + q\,(u_1 y_1 + u_2 y_2) = g \\ \Rightarrow & & u_1(y_1'' + p y_1' + q y_1) + u_2(y_2'' + p y_2' + q y_2) + u_1' y_1' + u_2' y_2' = g, \end{aligned}$$

but y_1 and y_2 are solution of

$$y'' + p y' + q y = 0,$$

so that

$$u_1' y_1' + u_2' y_2' = g.$$

Thus the two equations we have to solve for u_1 and u_2 are

$$\left\{ \begin{array}{rcl} u_1' y_1 + u_2' y_2 & = & 0 \\ u_1' y_1' + u_2' y_2' & = & g \end{array} \right. .$$

We'll solve these two linear algebraic equations for u_1' and u_2', then we'll integrate to get u_1 and u_2.

First, let's multiply the first equation by y_1' and the second equation by y_1. Subtracting the two resulting equations from each other, the terms with u_1' drop out and we get

$$\begin{aligned} & & u_2'(y_1 y_2' - y_2 y_1') = g y_1 \\ \Rightarrow & & u_2' W(y_1, y_2) = g y_1 \\ \Rightarrow & & u_2' = \frac{g y_1}{W(y_1, y_2)} \\ \Rightarrow & & u_2 = \int \frac{g y_1}{W(y_1, y_2)} dx. \end{aligned}$$

Note that y_1 and y_2 are by definition linearly independent, so that $W(y_1, y_2) \neq 0$.

Second, let's multiply the first equation by y_2' and the second equation by y_2. Subtracting the two resulting equations from each other (sounds familiar?), the terms with u_2' drop out and we get

$$u_1 = -\int \frac{gy_2}{W(y_1, y_2)} dx,$$

proceeding in an identical way to before. In both of these integrals, we don't care about the constants of integration since we just want to find one particular solution. As we've seen before, finding a different particular solution merely results in different constants c_1 and c_2 in the form of the general solution. Using $y_P = u_1 y_1 + u_2 y_2$, we get

$$\boxed{y_P = -y_1 \int \frac{y_2 g}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 g}{W(y_1, y_2)} dx.}$$

This is the most general form of the particular solution, for *any* given $g(x)$.

Example: Consider the equation

$$y'' - 5y' + 6y = 2e^x.$$

Here $g(x) = 2e^x$. Notice that this is an example where we could use either the method of undetermined coefficients or our new method of variation of parameters. No matter which method we use, we first have to solve the homogeneous problem.

Homogeneous solution: the characteristic equation is

$$\lambda^2 - 5\lambda + 6 = 0,$$

from which $\lambda_1 = 3$ and $\lambda_2 = 2$. Thus

$$y_1 = e^{3x}, \quad y_2 = e^{2x},$$

and

$$y_H = c_1 e^{3x} + c_2 e^{2x}.$$

Particular solution, using variation of parameters: first we compute the Wronskian:

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = e^{3x} 2e^{2x} - 3e^{3x} e^{2x} = -e^{5x}.$$

Then

$$\begin{aligned}
y_p &= -y_1 \int \frac{y_2 g}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 g}{W(y_1, y_2)} dx \\
&= -e^{3x} \int \frac{e^{2x} 2e^x}{-e^{5x}} dx + e^{2x} \int \frac{e^{3x} 2e^x}{-e^{5x}} dx \\
&= 2e^{3x} \int e^{-2x} dx - 2e^{2x} \int e^{-x} dx \\
&= 2e^{3x} \frac{e^{-2x}}{-2} - 2e^{2x} \frac{e^{-x}}{-1} \\
&= -e^x + 2e^x \\
&= e^x.
\end{aligned}$$

Particular solution, using undetermined coefficients: Given $g(x) = 2e^x$, we guess

$$y_P = Ae^x.$$

This guess is not contained in the homogeneous solution, so we don't need to modify it. Thus

$$y_P' = Ae^x \quad \text{and} \quad y_P'' = Ae^x.$$

Substituting these in gives

$$\begin{aligned} && Ae^x - 5Ae^x + 6Ae^x &= 2e^x \\ \Rightarrow && 2Ae^x &= 2e^x \\ \Rightarrow && A &= 1 \\ \Rightarrow && y_P &= e^x. \end{aligned}$$

Note that this is significantly faster than using variation of parameters.

General solution: the general solution is

$$y = c_1 e^{3x} + c_2 e^{2x} + e^x.$$

Example: Next, consider the equation

$$y'' + 9y = 9\sec^2 3x.$$

This example can't be done using the method of undetermined coefficients so variation of parameters is our only option.

Homogeneous solution: the characteristic equation is

$$\lambda^2 + 9 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 3i,$$

and

$$y_1 = \cos 3x, \quad y_2 = \sin 3x.$$

The homogeneous solution is

$$y_H = c_1 \cos 3x + c_2 \sin 3x.$$

Particular solution, using variation of parameters: first we compute the Wronskian:

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = (\cos 3x)(3\cos 3x) - (-3\sin 3x)(\sin 3x) = 3.$$

Then (using $u = 3x$ and $v = \cos u$)

$$
\begin{aligned}
y_p &= -y_1 \int \frac{y_2 g}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 g}{W(y_1, y_2)} dx \\
&= -\cos 3x \int \frac{9 \sec^2 3x}{3} \sin 3x dx + \sin 3x \int \frac{9 \sec^2 3x}{3} \cos 3x dx \\
&- \cos 3x \int \sec^2 u \sin u du + \sin 3x \int \sec^2 u \cos u du \\
&- \cos 3x \int \frac{-1}{v^2} dv + \sin 3x \int \sec u du \\
&= -\cos 3x \frac{1}{v} + \sin 3x \ln |\sec u + \tan u| \\
&- \cos 3x \frac{1}{\cos u} + \sin 3x \ln |\sec u + \tan u| \\
&- \cos 3x \frac{1}{\cos 3x} + \sin 3x \ln |\sec 3x + \tan 3x| \\
&= -1 + \sin 3x \ln |\sec 3x + \tan 3x|.
\end{aligned}
$$

General solution: the general solution is given by

$$y = c_1 \cos 3x + c_2 \sin 3x - 1 + \sin 3x \ln |\sec 3x + \tan 3x|.$$

As a final note, we should be careful when we start the method of variation of parameters: the form of the equation for which our solution formula is valid requires that the coefficient of y'' is one. Thus, if there is any other coefficient there originally, we have to divide the equation by it, so we can read of the correct form of $g(x)$.

Lecture 14. Mechanical vibrations

Problem set-up

As an application to second-order linear equations with constant coefficients, we go back to Newton's law:

$$F = ma.$$

Here F is the sum of the forces acting on the point particle of mass m, and a denotes the particle's acceleration. We'll consider the case of a particle suspended from a linear spring with spring constant k. The top of the spring could be moving in a prescribed way, and the particle is undergoing damping. You can think of damping as a consequence of dealing with a realistic spring (small damping) or maybe the whole process is taking place in a viscous bath. All of this is illustrated in Fig. 10.

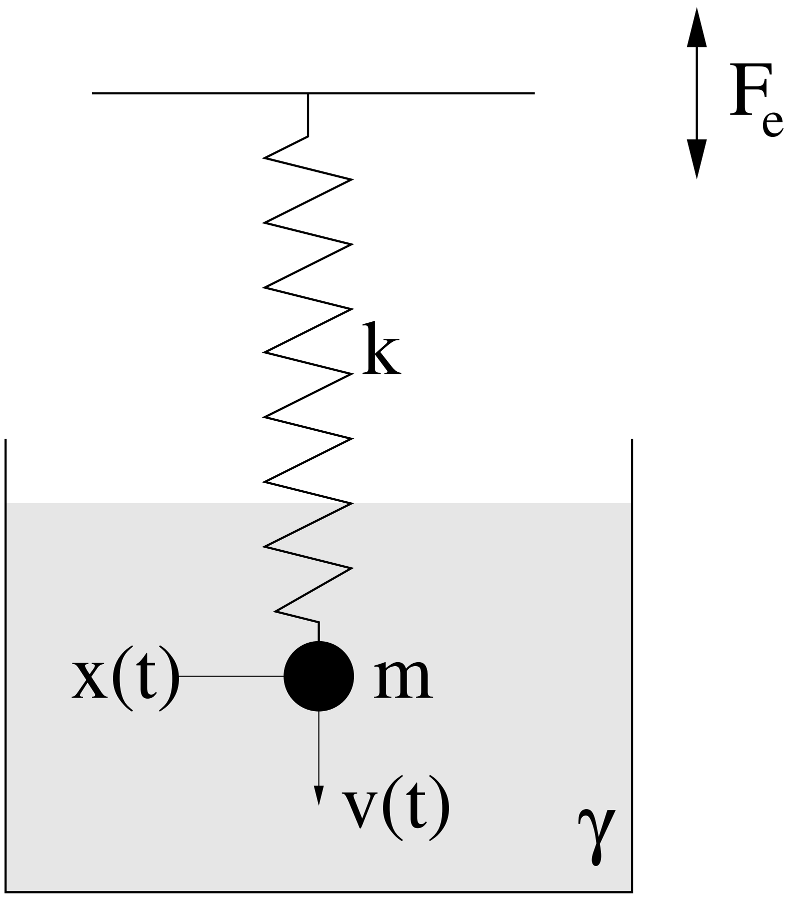

Figure 10: The set-up of our spring system.

So, what's our governing equation? We need to determine the explicit form of the total force. We have

$$F = F_{\text{spring}} + F_{\text{damper}} + F_{\text{external}}.$$

What are the functional forms of these different forces. The last one is given to us as

$$F_{\text{external}} = F_e(t),$$

some function of t. The other two are not much harder. The damping force is

$$F_{\text{damper}} = -\gamma v,$$

where v is the velocity of the particle, and γ is a constant damping rate. Note that this force has a negative sign: it opposes the motion. The last force is given by Hooke's law:

$$F_{\text{spring}} = -kx.$$

This force also comes with a minus sign. It is a restoring force: it pulls the particle back to its equilibrium position.

Putting all these together, we finally obtain

$$\boxed{mx'' + \gamma x' + kx = F_e(t).}$$

Here we've used that $a = x''$, $v = x'$: the acceleration and the velocity are the second, respectively first, time derivative of the position.

Unforced oscillations

If $F_e(t) \neq 0$ then the above differential equation is nonhomogeneous. As we've seen: whenever we're facing a nonhomogeneous problem, we should solve the homogeneous problem first. We'll get back to the nonhomogeneous problem when we talk about forced oscillations in the next lecture. Here we consider

$$mx'' + \gamma x' + kx = 0.$$

We refer to the motions predicted by this differential equation as *free* motions. Further, if $\gamma = 0$, the motion is undamped. Otherwise, if $\gamma > 0$, then the motion is damped.

We start by considering the characteristic equation:

$$m\lambda^2 + \gamma\lambda + k = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}.$$

Note that all of m, γ, and k are not allowed to be negative.

There are three possible cases:

1. $\gamma^2 > 4mk$: lots of damping. This is known as an *overdamped* spring.
2. $\gamma^2 = 4mk$: still a lot of damping, but less than overdamped. We call this a *critically damped* spring.
3. $\gamma^2 < 4mk$: a small amount of damping. This is known as the *underdamped* spring.

We'll spend most of our time studying the underdamped case. Note that the undamped spring is a special case of the underdamped spring.

Underdamped oscillations

If $\gamma^2 < 4mk$ then $4mk - c^2 > 0$, so that

$$\lambda_{1,2} = \frac{-\gamma \pm i\sqrt{4mk-\gamma^2}}{2m} = -\frac{\gamma}{2m} \pm i\omega,$$

where

$$\omega = \frac{4mk-\gamma^2}{2m}.$$

The general solution is given by

$$\begin{aligned} x &= c_1 e^{-\gamma t/2m}\cos\omega t + c_2 e^{-\gamma t/2m}\sin\omega t \\ &= e^{-\gamma t/2m}(c_1\cos\omega t + c_2\sin\omega t). \end{aligned}$$

Let's look at this solution in two different cases.

1. **The undamped spring: $\gamma = 0$.** In this case the exponential dissapears and

   $$x = c_1\cos\omega_0 t + c_2\sin\omega_0 t,$$

   with

   $$\omega_0 = \frac{\sqrt{4mk}}{2m} = \frac{\sqrt{mk}}{m} = \sqrt{\frac{k}{m}}.$$

   The parameter ω_0 is called the *natural frequency* of the system: it is the frequency the spring-particle system likes to oscillate at when no other forces (external, damping) are present. In order to completely determine the solution, we need initial conditions to specify the constants c_1 and c_2. Often it is useful to rewrite the solution formula in so-called *amplitude-phase* form. Let

   $$\begin{cases} c_1 &= A\cos\varphi \\ c_2 &= A\sin\varphi \end{cases}.$$

   Then

   $$A = \sqrt{c_1^2 + c_2^2}, \quad \tan\varphi = \frac{c_2}{c_1}.$$

   We have

   $$\begin{aligned} x &= A\cos\varphi\cos\omega_0 t + A\sin\varphi\sin\omega_0 t \\ &= A\cos(\omega_0 t - \varphi). \end{aligned}$$

   The new parameters A and φ are called the *amplitude* and the *phase* respectively, of the solution. We see that the solution is periodic with period

   $$T = \frac{2\pi}{\omega_0}.$$

   A plot of an undamped solution is shown in Fig. 11.

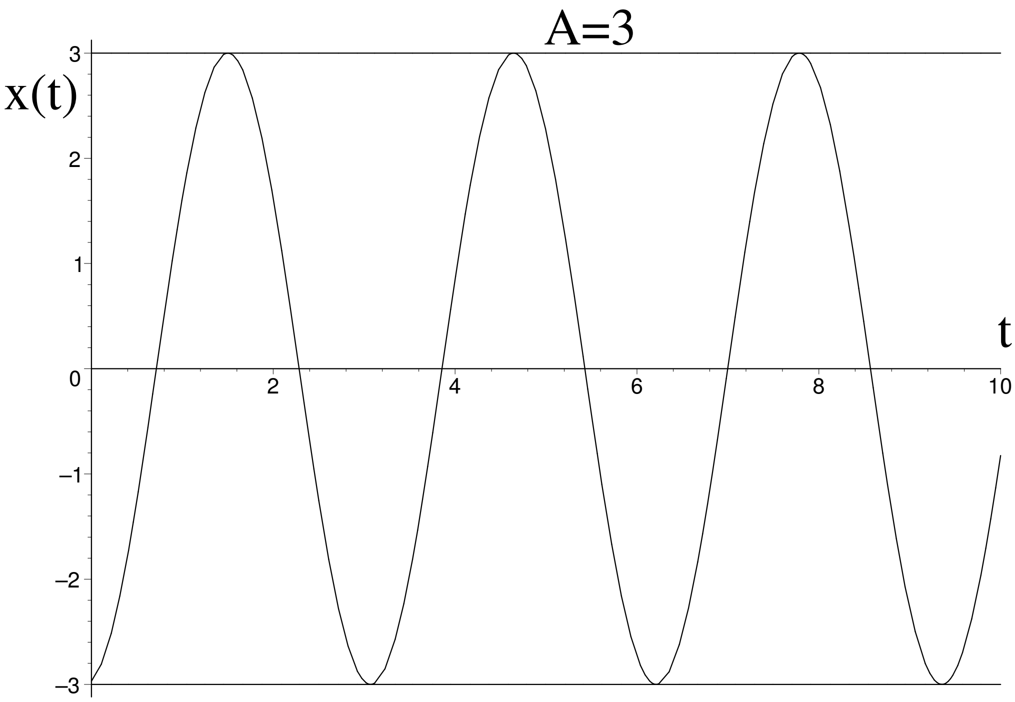

Figure 11: A solution of an undamped system: $x = 3\cos(2t - 3)$.

2. **The underdamped spring: $\gamma > 0$.**

   In the underdamped case with $\gamma > 0$ we have

$$\begin{aligned} x &= e^{-\gamma t/2m}(c_1 \cos \omega t + c_2 \sin \omega t) \\ &= A e^{-\gamma t/2m} \cos(\omega t - \varphi). \end{aligned}$$

   We see that the factor $Ae^{-\gamma t/2m}$ plays the role of a time-dependent amplitude. If the damping rate γ is small, then this amplitude factor will decay to zero, but at a slow rate. In this case the period of the solution is

$$T = \frac{2\pi}{\omega}.$$

   An underdamped solution is illustrated in Fig. 12.

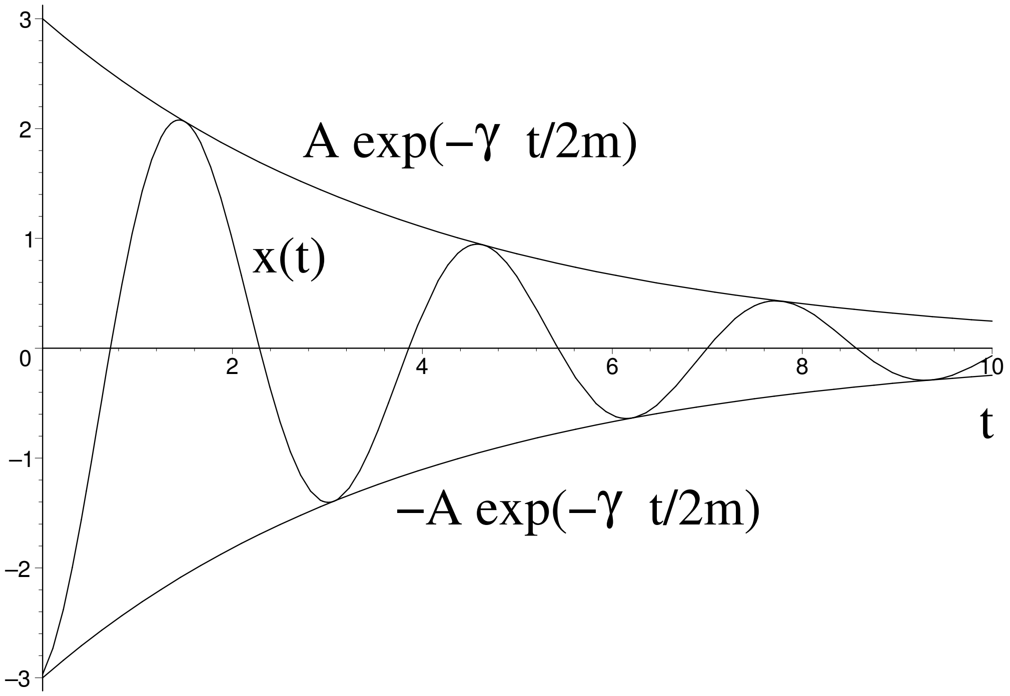

Figure 12: A solution of an underdamped system: $x = 3e^{-t/4}\cos(2t-3)$.

Lecture 15. Forced mechanical vibrations

Let's go back to the set-up at the beginning of the previous lecture. In this lecture we will include an external forcing term

$$F_e(t) = F_0 \cos \omega t.$$

Here F_0 and Ω are constants. This forcing represents a periodic moving up and down of the base of the spring system with constant amplitude F_0 and frequency Ω. Thus the differential equation we're looking at is

$$mx'' + \gamma x' + kx = F_0 \cos \Omega t.$$

This is a nonhomogeneous problem. That means we have to consider the homogeneous problem first. Fortunately, we've already done this in the previous lecture, so we get to use it here. As before, we'll break this up in different cases.

Remark: this same differential equation matters in a variety of different settings: mechanical systems such as springs, as discussed here; electrical systems with resistors, capacitors and solenoids, see below. In short, this differential equation is important to study in *any* setting where we encounter vibrations or oscillations.

1. No damping ($\gamma = 0$), no resonance ($\Omega \neq \omega_0$)

The differential equation is

$$\begin{aligned} & mx'' + kx = F_0 \cos \Omega t \\ \Rightarrow \quad & x'' + \omega_0^2 x = \frac{F_0}{m} \cos \Omega t, \end{aligned}$$

where $\omega_0^2 = k/m$ is the square of the natural frequency of the system.

(a) **The homogeneous solution** of this problem (see last lecture) is

$$x_H = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

and the frequency of these oscillations is ω_0.

(b) **The particular solution** can be found using the method of undetermined coefficients. We guess

$$x_p = A \cos \Omega t + B \sin \Omega t.$$

This guess looks like a good one, but we need to be careful: if $\Omega = \omega_0$, then the terms of the particular solution also appear in the homogeneous solution. In that case we need to multiply our guess by t and try again. We'll deal with this case separately later. So, for now: **assume that $\boldsymbol{\Omega} \neq \omega_0$**. In that case we substitute the above guess in the equation. With

$$x_p'' = -A\Omega^2 \cos \Omega t - B\Omega^2 \sin \Omega t,$$

we get

$$
\begin{aligned}
-A\Omega^2\cos\Omega t - B\Omega^2\sin\Omega t + \omega_0^2(A\cos\Omega t + B\sin\Omega t) &= \frac{F_0}{m}\cos\Omega t \\
\Rightarrow \qquad & \left\{\begin{array}{rcl} -A\Omega^2 + \omega_0^2 A &=& F_0/m \\ -B\Omega^2 + \omega_0^2 B &=& 0 \end{array}\right. \\
\Rightarrow \qquad & \left\{\begin{array}{rcl} A &=& \dfrac{F_0}{m(\omega_0^2 - \Omega^2)} \\ B &=& 0. \end{array}\right.
\end{aligned}
$$

We see immediately that there are problems with the solution if we were to allow $\Omega = \omega_0$. Good thing we excluded this! The particular solution is

$$x_p = \frac{F_0}{m(\omega_0^2 - \Omega^2)}\cos\Omega t.$$

(c) **The general solution** is given by

$$x = x_H + x_p = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{F_0}{m(\omega_0^2 - \Omega^2)}\cos\Omega t.$$

At this point, c_1 and c_2 may be determined from the initial conditions.

Let's impose some special initial conditions. These aren't really essential, but they make the calculations a bit easier. Let

$$x(0) = 0, x'(0) = 0.$$

You'll easily check that the corresponding solution is given by

$$x = \frac{F_0}{m(\omega_0^2 - \Omega^2)}(\cos\Omega t - \cos\omega_0 t).$$

(You checked this, right? Otherwise go back and do it NOW!) Using a trig identity (check this too!) this solution is rewritten as

$$x = \frac{2F_0}{m(\omega_0^2 - \Omega^2)}\sin\omega_1 t\sin\omega_2 t,$$

where

$$\omega_1 = \frac{\omega_0 - \Omega}{2}, \quad \omega_2 = \frac{\omega_0 + \Omega}{2}.$$

Assume that Ω is close to ω_0 (but not equal, otherwise the above result is not valid, remember?) then ω_1 is close to zero, which means that the factor $\sin\omega_1 t$ is a function with a frequency that is much smaller than that of $\sin\omega_2 t$. We can trivially rewrite our solution as

$$x = \left(\frac{2F_0}{m(\omega_0^2 - \Omega^2)}\sin\omega_1 t\right)\sin\omega_2 t = U(t)\sin\omega_2 t,$$

where

$$U(t) = \frac{2F_0}{m(\omega_0^2 - \Omega^2)} \sin \omega_1 t$$

is interpreted as a time-dependent amplitude: it is a function that is changing much slower than $\sin \omega_2 t$. This time-dependent amplitude is itself oscillating in time, but it takes a lot longer for it to come around. The kind of pattern we get is illustrated in Fig. 13.

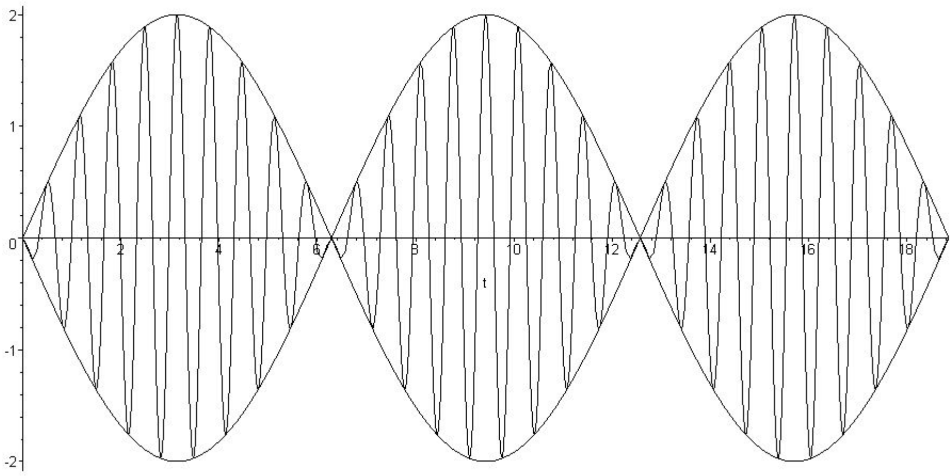

Figure 13: The beats phenomenon with $\Omega = 10$, $\omega_0 = 9$.

Such a signal is called a modulated wave, and the phenomenon observed is that of **beats**: there are two frequencies in this problem. The first frequency is the slow one, which governs the *modulation* of the amplitude. The second frequency is that of the underlying *carrier wave*, *i.e.*, the fast oscillations.

2. No damping ($\gamma = 0$), resonance ($\Omega = \omega_0$)

Let's look at one of the special cases we skipped. The solution given above is not valid when $\Omega = \omega_0$. What happens in this case?

The differential equation is

$$\begin{aligned} & mx'' + kx = F_0 \cos \omega_0 t \\ \Rightarrow \qquad & x'' + \omega_0^2 x = \frac{F_0}{m} \cos \omega_0 t. \end{aligned}$$

(a) **The homogeneous solution** of this problem is the same as before:

$$x_H = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

(b) **The particular solution** can be found using the method of undetermined coefficients. We guess

$$x_p = A \cos \omega_0 t + B \sin \omega_0 t.$$

This guess is no longer valid, since both terms of our guess occur in the homogeneous solution. This implies we need to multiply our guess by t and try again. We get

$$\begin{aligned}
& x_p = At\cos\omega_0 t + Bt\sin\omega_0 t, \\
\Rightarrow\quad & x_p' = A\cos\omega_0 t + B\sin\omega_0 t - A\omega_0 t\sin\omega_0 t + B\omega_0 t\cos\omega_0 t, \\
\Rightarrow\quad & x_p'' = -2A\omega_0\sin\omega_0 t + 2B\omega_0\cos\omega_0 t - A\omega_0^2 t\cos\omega_0 t - B\omega_0^2 t\sin\omega_0 t.
\end{aligned}$$

Substituting this in the equation, we obtain

$$\begin{aligned}
& -2A\omega_0\sin\omega_0 t + 2B\omega_0\cos\omega_0 t - A\omega_0^2 t\cos\omega_0 t - B\omega_0^2 t\sin\omega_0 t + \\
& \qquad \omega_0^2\left(At\cos\omega_0 t + Bt\sin\omega_0 t\right) = \frac{F_0}{m}\cos\omega_0 t \\
\Rightarrow\quad & -2A\omega_0\sin\omega_0 t + 2B\omega_0\cos\omega_0 t = \frac{F_0}{m}\cos\omega_0 t \\
\Rightarrow\quad & \left\{\begin{array}{rcl} -2A\omega_0 & = & 0 \\ 2B\omega_0 & = & \dfrac{F_0}{m} \end{array}\right. \\
\Rightarrow\quad & \left\{\begin{array}{rcl} A & = & 0 \\ B & = & \dfrac{F_0}{2m\omega_0}. \end{array}\right.
\end{aligned}$$

The particular solution is

$$x_p = \frac{F_0}{2m\omega_0} t\sin\omega_0 t.$$

(c) **The general solution** is given by

$$x = x_H + x_p = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{F_0}{2m\omega_0} t\sin\omega_0 t.$$

At this point, c_1 and c_2 may be determined from the initial conditions.

Let's think about this solution. After a significant time, the particular solution will give us the most important part, as it's linearly increasing in time, whereas the homogeneous solution is just oscillating from here to oblivion. So, what does this particular solution look like? It's plotted in Fig. 14. You observe that the amplitude of the solution is linearly growing in time. This phenomenon is called *resonance*. It occurs whenever we force a system at its natural frequency. Resonance is one of the important elementary processes in all kinds of physical systems. You may imagine that this is not necessarily a good thing in applications: if we force the spring to oscillate at higher and higher amplitudes, it may eventually break! This gives us another way to think about the natural frequency of the system: it is the frequency that if we use it to force the system results in the system oscillating more and more wildly, eventually leading to breakdown, unless we have a way to prevent it. Preventing it is the subject of the next case.

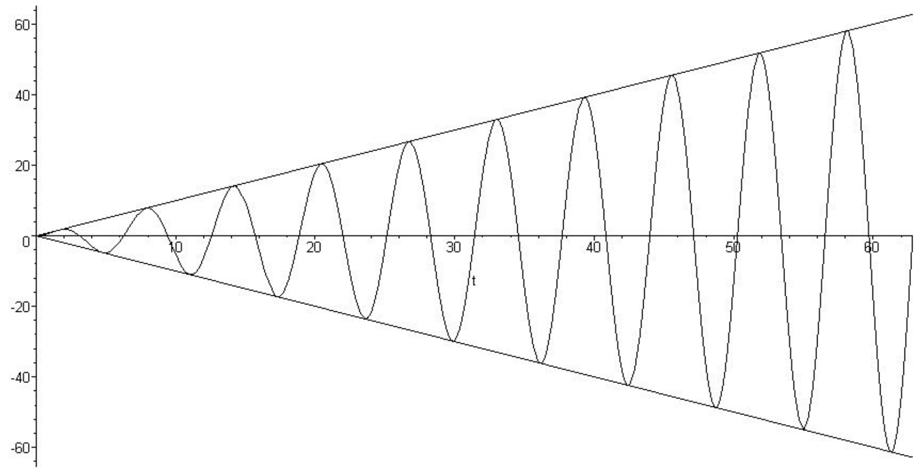

Figure 14: The phenomenon of resonance with $\omega_0 = 1$.

3. Damping ($\gamma \neq 0$)

Let's look at what happens when we include the effects of damping. In any realistic system some amount of damping will be present. Sometimes, its effects are so minuscule they may be ignored. In other cases, they may dominate.

The differential equation is

$$mx'' + \gamma x' + kx = F_0 \cos \omega_0 t.$$

(a) We've seen how to find the **homogeneous solution** of this problem in the previous lecture: assuming that we're dealing with subcritical damping we have

$$x_H = e^{-\gamma t/2m} \left(c_1 \cos \omega t + c_2 \sin \omega_0 t \right),$$

where $\omega = \sqrt{4km - \gamma^2}/2m$. Note that by assuming subcritical damping we've let $\gamma^2 < 4km$. As we've seen this corresponds to a damped oscillation. Thus, no matter what the particular solution is, or what the initial conditions are, we have

$$\lim_{t \to \infty} x_H = 0.$$

This implies that, if we wait sufficiently long, all the important information about the general solution is contained in the particular solution! So, what are we waiting for? Let's find it!

(b) **The particular solution** can be found using the method of undetermined coefficients, as before. We guess

$$x_p = A \cos \Omega t + B \sin \Omega t.$$

After substituting this guess in the equation and equating the coefficients of sine and cosine, and doing some algebra, we get (check this!):

$$\begin{cases} A = \dfrac{\frac{F_0}{m}(\omega_0^2 - \Omega^2)}{(\omega_0^2 - \Omega^2)^2 + \gamma_0^2\Omega^2}, \\ \\ B = \dfrac{\frac{F_0}{m}\gamma\Omega}{(\omega_0^2 - \Omega^2)^2 + \gamma_0^2\Omega^2}, \end{cases}$$

where $\gamma_0 = \gamma/m$. We see that the particular solution is always bounded as $t \to \infty$. Even if we were to have $\Omega = \omega_0$, or $\Omega = \omega$, the particular solution we've constructed works just fine. Thus there's never a danger of the amplitude of the particular solution exploding on us, as there was in the resonant case without damping.

(c) **The general solution** is given by

$$x = x_H + x_p = e^{-\gamma t/2m}\left(c_1 \cos \omega t + c_2 \sin \omega_0 t\right) + A \cos \Omega t + B \sin \Omega t,$$

where A and B are given by the expressions above. At this point, c_1 and c_2 may be determined from the initial conditions.

Let's think about this solution. After a significant time, the particular solution will give us the only important part, as it's not decaying in time, whereas the homogeneous solution is. On the other hand, the particular solution is just an oscillation. What can we say about it? One of the most important aspects of an oscillation is its amplitude. For the particular solution here, that amplitude is given by (do I need to say it: Check it!)

$$\sqrt{A^2 + B^2} = \frac{F_0/m}{\sqrt{\gamma_0^2\Omega^2 + (\omega_0^2 - \Omega^2)^2}}.$$

It is clear from this formula that the magnitude of the response of the system depends a lot on the parameters of the input forcing. To quantify that, we rewrite the above as

$$\frac{m}{F_0}\sqrt{A^2 + B^2}\,\omega_0^2 = \frac{1}{\sqrt{\frac{\gamma_0^2}{\omega_0^2}\frac{\Omega^2}{\omega_0^2} + \left(1 - \frac{\Omega^2}{\omega_0^2}\right)^2}}.$$

This expression is used to plot the amplitude response graph, shown in Fig. 15. This figure shows the scaled (by a factor m/F_0) amplitude of the response, as a result of forcing the system with frequency Ω (in units of ω_0), for different values of the normalized damping γ_0/ω_0. We see that for no damping, there is a vertical asymptote at $\Omega/\omega_0 = 1$, as expected. For non-zero damping, there is still a maximum in the amplitude near $\Omega/\omega_0 = 1$. Thus, if we want to get a lot from a little (and who doesn't?), we should force the system with a frequency that is close to its natural frequency, as this will maximize the amplitude of the output response.

Microwaves work on this principle: the microwave operates in the microwave regime (gee, coincidence?), which is close to the natural frequency for water molecules. Water is the main ingredient in any food. As a result of the microwave forcing, the water molecules vibrate a lot, giving off a lot of heat due to friction. It is this heat that warms your food.

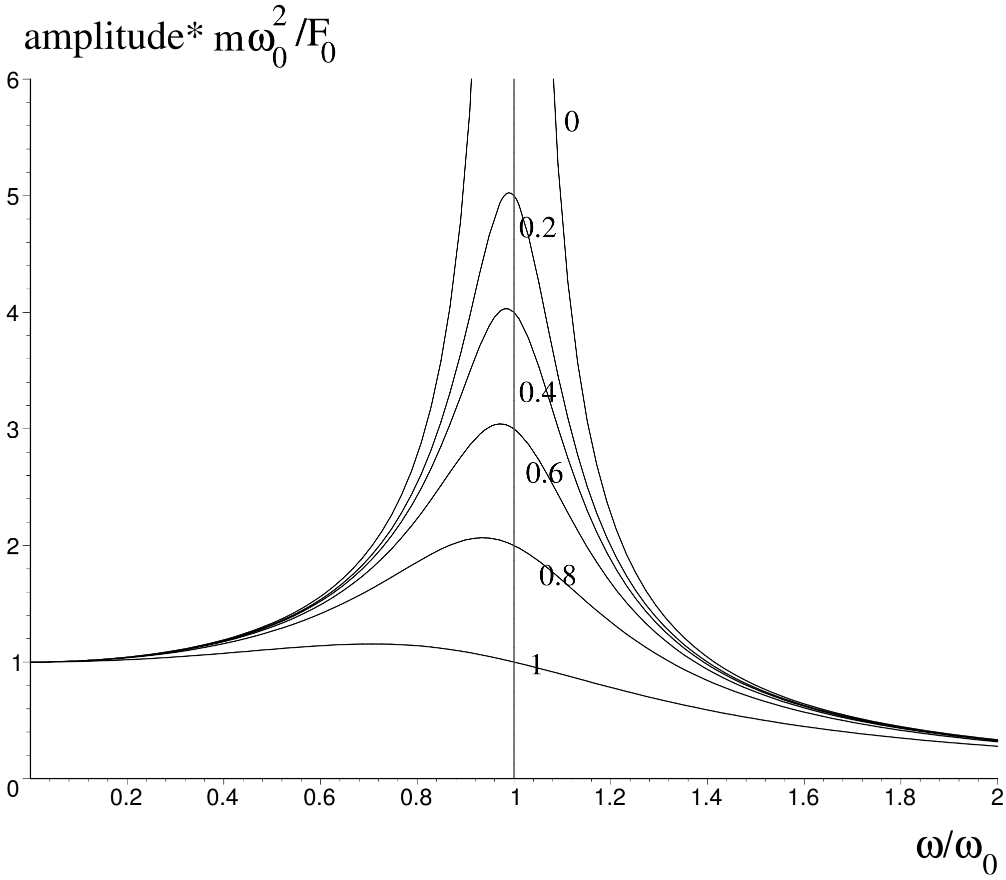

Figure 15: The amplitude-response graph for various values of γ_0/ω_0.

Lecture 16. Systems and linear algebra I: introduction

Systems of equations

Let's take another look at the second-order equation

$$y'' + p(x)y' + q(x)y = g(x).$$

We can rewrite this as a few first-order equations. Let

$$\begin{cases} y_1 &= y, \\ y_2 &= y', \end{cases}$$

then

$$\begin{cases} y_1' &= y' = y_2, \\ y_2' &= y'' = -p(x)y' - q(x)y + g(x), \\ &= -p(x)y_2 - q(x)y_1 + g(x), \end{cases}$$

or

$$\begin{cases} y_1' &= y_2, \\ y_2' &= -p(x)y_2 - q(x)y_1 + g(x). \end{cases}$$

This is a *system* of first-order equations. The word *system* refers to the fact that there's more than one equation we have to solve. The above system has *dimension* two, which means that there's two equations to solve, and two functions (y_1 and y_2) to solve for. We can have systems of any number of dimensions, as we'll see in what follows. Further, similarly to what we did above, every differential equation of any order can be rewritten as a system of first-order differential equations. Even better, every system of equations of arbitrary order can be rewritten as a system of first-order equations. Let's do another example. Such systems of higher-order equations often arise in their own right in applications. Just think of Newton's law applied to a multiparticle situation.

Example: Consider the third-order equation

$$y''' + 2y' + 5y = 7.$$

Since this equation is of third order, we introduce three variables:

$$\left\{ \begin{array}{rcl} y_1 &=& y, \\ y_2 &=& y', \\ y_3 &=& y'', \end{array} \right.$$

Then

$$\left\{ \begin{array}{rcl} y_1' &=& y' = y_2, \\ y_2' &=& y'' = y_3, \\ y_3' &=& y''' = -2y' - 5y + 7 = -2y_2 - 5y_1 + 7. \end{array} \right.$$

Thus our corresponding first-order system is

$$\left\{ \begin{array}{rcl} y_1' &=& y_2, \\ y_2' &=& y_3, \\ y_3' &=& -2y_2 - 5y_1 + 7. \end{array} \right.$$

Let's do an example with a higher-order system.

Example: Consider the system

$$\left\{ \begin{array}{rcl} y''' + y &=& y' - y^2, \\ u'' + u &=& y. \end{array} \right.$$

Here we have a system of two equations: one is of third order, the other one is of second order. How do we write this as a system of first-order equations? We'll need five variables:

$$\left\{ \begin{array}{rcl} y_1 &=& y, \\ y_2 &=& y', \\ y_3 &=& y'', \\ u_1 &=& u, \\ u_2 &=& u'. \end{array} \right.$$

Then

$$\left\{ \begin{array}{rcl} y_1' &=& y' = y_2, \\ y_2' &=& y'' = y_3, \\ y_3' &=& y''' = -y + u' - y^2 = -y_1 + u_2 - y_1^2, \\ u_1' &=& u' = u_2, \\ u_2' &=& u'' = -u + y = -u_1 + y_1. \end{array} \right.$$

Thus our first order system is

$$\begin{cases} y_1' &= y_2, \\ y_2' &= y_3, \\ y_3' &= y_1 + u_2 - y_1^2, \\ u_1' &= u_2, \\ u_2' &= -u_1 + y_1. \end{cases}$$

Note that this system is nonlinear (because of the y_1^2 term). This is to be expected, since the originating system is also nonlinear. This is a fifth-order system.

Now we consider the most general first-order system that is linear. This has to be of the form

$$\begin{cases} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + b_1, \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + b_2, \\ y_3' &= a_{31}y_1 + a_{32}y_2 + \cdots + a_{3n}y_n + b_3, \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + b_n. \end{cases}$$

Here the coefficients a_{11}, a_{12} *etc* are allowed to be functions of our independent variable (call it t). So are the functions $b_1, \ldots, b_n$. This is an n-th order system. The unknown functions are $y_1, \ldots, y_n$. Let's collect these in a list that we'll call y, which we'll write vertically as

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

We call a vertical list like this an *n-dimensional column vector*. If we had written the list horizontally, it'd be an *n-dimensional row vector*. We can group the functions $b_1, \ldots, b_n$ in a similar vector:

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Organizing the coefficient functions $a_{11}, a_{12}, \ldots$ is a bit more complicated: we have to keep track of which ones belong to the first equation, to the second equation and so on. Further, we have to see which coefficients belong to which unknown function. So we won't just put all the $a_{11}, a_{12}, \ldots$ in a long list. Rather, we put them in a rectangular table: all the ones from the first equation go in the first row. The ones from the second equation in the second row, and so on. Similarly, the ones multiplying y_1 we'll put in the first column, the ones multiplying y_2 in the second column, and so on. We denote the whole table by A. Thus

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

Such a rectangular table of entries is referred to as a *matrix*. The above matrix has n rows and n columns. Therefore it's called a matrix of size $n \times n$. Since the number or rows equals the number of columns, we call A a square matrix. We'll see nonsquare matrices later as well. Actually, you've already seen them: both y and b are examples of matrices of size $n \times 1$. Thus column (and row) vectors are a special type of matrix.

Introducing all of this notation allows us to rewrite the system in shorthand as

$$y' = Ax + b.$$

Now, you have to admit: even if that's all it's good for, that's pretty good. That's a lot less writing for sure. The beauty of the whole thing is that this is not all it's good for: it's good for far more! That's the topic of *linear algebra*. Linear algebra is very useful, also in studying differential equations. In the remainder of this lecture, and in the next few lectures, we'll go over the essentials of linear algebra that we need in the rest of this course.

A *matrix* is a rectangular array of entries, arranged in row and columns. Matrices are the fundamental objects in linear algebra. Let's figure out how we work with them.

We can multiply matrices with vectors: from the above definitions, you already know how to do this:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ \vdots \\ a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{pmatrix}.$$

Thus, the result of multiplying a matrix of size $n \times n$ with a column vector of size n is a new column vector of size n. For this multiplication to work, the number of columns of the matrix needs to be equal to the size of the column vector. Before we say more about multiplication, let's say more about even more elementary properties of matrices.

Elementary properties of matrices

Let's use the following matrices in our examples:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 5 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Note that A is a square matrix of size 3×3, B is a nonsquare matrix of size 2×3, whereas C is a column vector of size 3. Alternatively, C is a nonsquare matrix of size 3×1.

- **Rows:** the first row of A is $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$. The second row of B is $\begin{pmatrix} 7 & 5 & -2 \end{pmatrix}$. The last row of C is $\begin{pmatrix} 3 \end{pmatrix}$.
- **Columns:** The third column of A is

$$\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix},$$

and the second column of B is

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

- **Transpose:** The transpose of a matrix is the same matrix, but with the rows and columns switched: what used to be the first row in now the first columns, *etc.* The transpose of a matrix is denoted with a super-index T:

$$A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

  Similary,

$$C^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \quad B^T = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 5 & -2 \end{pmatrix}^T = \begin{pmatrix} 1 & 7 \\ 2 & 5 \\ 3 & -2 \end{pmatrix}.$$

- **Equality:** two matrices are equal if all their entries are equal. Note that this can only happen if the matrices are the same size, *i.e.*, the two matrices have the same number of rows, and the same number of columns.

- **Addition, subtraction:** We can only add or subtract matrices of equal dimensions. Then $A \pm B$ is the matrix with as entries

$$(A \pm B)_{ij} = (A)_{ij} \pm (B)_{ij}.$$

  In other words, addition and subtraction are done entry by entry. As an example:

$$\begin{pmatrix} 2 & 3 & 5 \\ 7 & 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 2 & 1 \\ 0 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 6 \\ 7 & 5 & 0 \end{pmatrix}.$$

  You can easily check that, as a consequence of our definition, the following properties hold:

$$\begin{aligned} A + (B + C) &= (A + B) + C && \text{(Associativity)}, \\ A + B &= B + A && \text{(Commutativity)}. \end{aligned}$$

Multiplication of matrices

For starters, we can only multiply the matrices A and B is the number of columns of A equals the number of rows of B. If so, then the resulting matrix AB has the same number of rows as A and the same number of columns as B:

$$\underset{n \times m}{A} \quad \underset{m \times k}{B} \quad = \quad \underset{n \times k}{AB}.$$

Thus the result of multiplying a matrix of size $n \times m$ with one of size $m \times k$ is a matrix of size $n \times k$. It's entry at position ij is given by

$$\begin{aligned} (AB)_{ij} &= \sum_{r=1}^{m} A_{ir} B_{rj} \\ &= (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B) \\ &= \text{scalar product of the the } i\text{-th row of } A \text{ with the } j\text{-th row of } B. \end{aligned}$$

You may have seen the scalar product before. If not, or else as a reminder, the scalar product of two m-dimensional vectors $v^{(1)}$ and $v^{(2)}$ is

$$v^{(1)} \cdot v^{(2)} = v_1^{(1)} v_1^{(2)} + v_2^{(1)} v_2^{(2)} + \cdots + v_m^{(1)} v_m^{(2)}.$$

Let's do a few examples.

Example: Let

$$A = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

We cannot compute AB, since the A is of dimension 1×3 and B has dimension 2×1. Note that we could compute BA though. Why don't you do it as an exercise?

Example: Let

$$A = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 \\ 2 \\ 7 \end{pmatrix}.$$

These two matrices can be multiplied: we know that AB will be a 1×1 matrix. We get

$$\begin{aligned} AB &= 1 \times 1 \text{ matrix} \\ &= (\text{first row of } A \cdot \text{first column of } B) \\ &= (1 * 0 + 3 * 2 + 5 * 7) \\ &= (0 + 6 + 35) \\ &= (41). \end{aligned}$$

The result is the 1×1 matrix with as sole entry 41.

Example: Now, a little bit harder. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

In this case we can compute either AB or BA. Either one will be a 2×2 matrix again. Let's do both.

$$\begin{aligned} AB &= \begin{pmatrix} \text{row 1 of } A \cdot \text{column 1 of } B & \text{row 1 of } A \cdot \text{column 2 of } B \\ \text{row 2 of } A \cdot \text{column 1 of } B & \text{row 2 of } A \cdot \text{column 2 of } B \end{pmatrix} \\ &= \begin{pmatrix} -2 + 4 & 1 + 0 \\ -6 + 8 & 3 + 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
BA &= \begin{pmatrix} \text{row 1 of } B \cdot \text{column 1 of } A & \text{row 1 of } B \cdot \text{column 2 of } A \\ \text{row 2 of } B \cdot \text{column 1 of } A & \text{row 2 of } B \cdot \text{column 2 of } A \end{pmatrix} \\
&= \begin{pmatrix} -2+3 & -4+4 \\ 2+0 & 4+0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}.
\end{aligned}$$

Note that these are not equal! This may come as a suprise. The matrix multiplication of A with B is usually not equal to that of B with A. For one, it is possible for AB and BA to have different dimensions! Heck: one of them may be defined whereas the other one is not. Even if both are defined, the numbers typically (but not always) come out to be different! Hmmm. Perhaps you don't like this. Oh well. You'll live.

Example: We won't be stopped: let's do a 3×3 example: let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}.$$

Then

$$\begin{aligned}
AB &= \begin{pmatrix} 1+2+3 & 0+4+6 & 0+0+9 \\ 0+4+5 & 0+8+10 & 0+0+15 \\ 0+0+6 & 0+0+12 & 0+0+18 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 10 & 9 \\ 9 & 18 & 15 \\ 6 & 12 & 18 \end{pmatrix}.
\end{aligned}$$

Are we getting the hang of this?

Example: Let

$$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 \end{pmatrix}.$$

Then AB is a 2×2 matrix:

$$AB = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}.$$

You can easily check that, as a consequence of our definition, the following properties hold:

$$\begin{aligned}
A(BC) &= (AB)C && \text{(Associativity)}, \\
(A+B)C &= AC + BC && \text{(Distributivity)}.
\end{aligned}$$

On the other hand, as we've just discussed in the examples:

$$AB \neq BA,$$

except in rare cases.

Lastly, we need to define a special matrix which will be convenient later:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

This matrix has zeros everywhere, except on the diagonal, where it has 1's. Often we omit the index n, which denotes the size of this matrix, which is called the *identity matrix*. The reason it is called this is that

$$AI = A = IA.$$

Thus, multiplication with the identity matrix has no effect.

Lecture 17. Systems and linear algebra II: RREF

We've learnt a lot about matrices. What can we do with them? In this lecture, we'll see how to use matrices to solve linear algebraic equations. To this end, we introduce the following operations:

- $M_k(\alpha)$: this operation multiplies the k-th row of our matrix by $alpha$,
- P_{ij}: this operation switches rows i and j of our matrix,
- $E_{ij}(\alpha)$: this operation adds row i multiplied by α to row j.

Example: Let's see how these work. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Then

$$P_{23}A = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix},$$

where we've switched rows 2 and 3. Similarly,

$$M_1(2)A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

and we've multiplied the first row by 2. Lastly,

$$E_{13}(2)A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7+2*1 & 8+2*2 & 9+2*3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 12 & 15 \end{pmatrix}.$$

We've simply added twice the first row to the third row.

The idea behind these operations is that we can use them to bring a matrix to its RREF (row-reduced echelon form). Now, what on earth is the RREF?

Definition (RREF): a matrix is in row-reduced echelon form if:
(1) the first non-zero element in any row is 1 (this is called a leading 1),
(2) all elements in the same column as a leading 1 are 0,
(3) a leading 1 in a row is to the right of the leading 1's in all rows above it,

(4) if there are any rows with all zeros, they are at the bottom.

Example: Let's see how we can bring a matrix to its RREF. Let's use our familiar matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \overset{E_{12}(-4),\ E_{13}(-7)}{\longrightarrow} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$
$$\overset{M_2(-1/3),\ M_3(-1/6)}{\longrightarrow} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$
$$\overset{E_{21}(-2),\ E_{23}(-1)}{\longrightarrow} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

which is RREF. Thus

$$\mathrm{RREF}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Why is this a useful thing to do? Why do we want to bring a matrix to its RREF?

Gaussian elimination: solving systems of linear equations

Suppose we want to solve the system of equations

$$\begin{cases} x_1 + x_2 - 10x_3 &= 1 \\ -x_1 + 10x_3 &= -2 \\ x_1 + 4x_2 - 5x_3 &= -1 \end{cases}.$$

We can write this in matrixform as

$$Ax = b,$$

where

$$A = \begin{pmatrix} 1 & 1 & -10 \\ -1 & 0 & 10 \\ 1 & 4 & -5 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Now we consider the *augmented matrix*:

$$(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & -10 & 1 \\ -1 & 0 & 10 & -2 \\ 1 & 4 & -5 & -1 \end{array}\right).$$

This augmented matrix represents the system of equations we want to solve: the first row represents the first equation, the second row the second equation and so on. Similarly, the first column corresponds to the variable x_1, *etc.* The last column corresponds to the right-hand side of the original equations. Thus, every row corresponds to an equation, and every column (except the last one) corresponds to a variable.

Now we see what the elementary row operations do, in terms of the underlying equations:

- P_{ij}: this switches the order of the equations i and j,
- $M_i(\alpha)$: this multiplies the i-th equation by α, and
- $E_{ij}(\alpha)$: this adds multiples of equations.

The important observation about all of these operations is that none of them affect the solutions of the equations we're trying to solve. Thus, if we use these operations to bring a matrix to its RREF, we can solve the equations corresponding to that RREF and the solutions will be the same. That's dandy: the solutions for the equations corresponding to the RREF are A LOT easier to find than those of the original equations!

Example: Let's use our previous augmented matrix:

$$\begin{aligned}
\left(\begin{array}{rrr|r} 1 & 1 & -10 & 1 \\ -1 & 0 & 10 & -2 \\ 1 & 4 & -5 & -1 \end{array}\right) &\overset{E_{12}(1),\ E_{13}(-1)}{\longrightarrow} \left(\begin{array}{rrr|r} 1 & 1 & -10 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 3 & 5 & -2 \end{array}\right) \\
&\overset{E_{21}(-1),\ E_{23}(-3)}{\longrightarrow} \left(\begin{array}{rrr|r} 1 & 0 & -10 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 5 & 1 \end{array}\right) \\
&\overset{M_3(1/5)}{\longrightarrow} \left(\begin{array}{rrr|r} 1 & 0 & -10 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/5 \end{array}\right) \\
&\overset{E_{31}(10)}{\longrightarrow} \left(\begin{array}{rrr|r} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/5 \end{array}\right),
\end{aligned}$$

from which it follows immediately (remember: columns correspond to variables; the last column corresponds to the right-hand side) that

$$x_1 = 4, \quad x_2 = -1, \quad x_3 = 1/5.$$

I'm not saying that for any given system, this is the most efficient way to solve it, but it most definitely is the most systematic way. And there's value in that: this is the way computers solve linear systems (with minor modifications). It's tedious, but braindead. Sounds like a good idea!

Example: Let's do another example. This time, we'll consider a system of equations that depends on a parameter. We'll investigate how the solutions depend on that parameter. Our system is

$$\left\{\begin{array}{rcl} x_1 - x_2 + x_3 &=& -1 \\ x_2 - x_3 &=& 3 \\ x_1 + x_2 - x_3 &=& a \end{array}\right.,$$

where a is a real parameter. The augmented matrix is

$$\left(\begin{array}{rrr|r} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 3 \\ 1 & 1 & -1 & a \end{array}\right).$$

RREF, here we come!

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 3 \\ 1 & 1 & -1 & a \end{array}\right) \overset{E_{13}(-1)}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 2 & -2 & a+1 \end{array}\right)$$

$$\overset{E_{21}(1),\ E_{23}(-2)}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & a-5 \end{array}\right).$$

We're not at our desired RREF yet, but we have to consider two different cases.

Case 1. $a = 5$ Then we have

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & a-5 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This matrix is RREF. The underlying equations are:

$$\left\{\begin{array}{rcl} x_1 & = & 2 \\ x_2 - x_3 & = & 3 \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{rcl} x_1 & = & 2 \\ x_2 & = & 3 + x_3 \end{array}\right. .$$

We can write these solutions in vectorform:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ x_3 + 3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

We see that there are an infinite number of solutions in this case: we get to choose whatever value we want for x_3, and all such values result in solutions. Notice that in this case, we obtained no information about our solution from the third equation, which is why we got to choose one of the variables.

Case 2. $a \neq 5$ Then our matrix reduces to

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & a-5 \end{array}\right).$$

The equation corresponding to the third row says

$$0 = a - 5,$$

which is not true, since $a \neq 5$. Thus, in this case, there are no solutions!

Lecture 18. Systems and linear algebra III: linear dependence and independence

Let's get right to it:

Definition (Linear (in)dependence): the vectors $x^{(1)}$, $x^{(2)}$, ..., $x^{(n)}$ are called linearly dependent if there exist constants k_1, k_2, ..., k_n (not all zero) such that

$$k_1x^{(1)} + k_2x^{(2)} + \ldots + k_nx^{(n)} = 0.$$

In other words: at least one of the vectors can be written as a linear combination of the others. If the vectors are not linearly dependent, they are called linearly independent.

If the vectors are linearly dependent, we are often interested in finding out the values of the constants k_1, k_2, ..., k_n to find out what relationship exists between the vectors.

Example: Are

$$x^{(1)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad x^{(3)} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

linearly dependent or independent? Or: can we find constants k_1, k_2 and k_3 (not all zero) such that

$$k_1x^{(1)} + k_2x^{(2)} + k_3x^{(3)} = 0.$$

Let's write this last equation out in more detail. We have

$$k_1 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + k_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0,$$

or rewritten as a regular system of equation:

$$\left\{ \begin{array}{rcl} 2k_1 - k_3 & = & 0 \\ k_1 + k_2 + 2k_3 & = & 0 \\ 3k_1 + k_2 + k_3 & = & 0 \end{array} \right. .$$

Let's rewrite this system in matrixform. We get

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is clarifying: the matrix we have to row-reduce is nothing but the matrix where we put the first vector in the first column, the second vector in the second column, and so on. We could work with the augmented matrix and add a column of zeros at the end, but that doesn't do any good for anyone involved so we might as well leave it off. Let's start row-reducing.

$$
\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 3 & 1 & 1 \end{pmatrix} \overset{P_{12}}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ 3 & 1 & 1 \end{pmatrix}
$$

$$
\overset{E_{12}(-2),\ E_{13}(-3)}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & -2 & -5 \end{pmatrix}
$$

$$
\overset{M_2(-1/2)}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 5/2 \\ 0 & -2 & -5 \end{pmatrix}
$$

$$
\overset{M_2(-1/2)}{\longrightarrow} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{pmatrix},
$$

from which it follows that

$$
\begin{cases} k_1 - \dfrac{1}{2}k_3 & = & 0 \\ k_2 + \dfrac{5}{2}k_3 & = & 0. \end{cases}
$$

It follows that we can choose k_3 to be whatever we want. Now, I don't know what you want, but if you just manage to convince yourself to pick $k_3 \neq 0$, everyone's happy. Given that we're in this for the perennial pursuit of happiness, we'll proceed this way. We'll even be smart about it: to avoid fractions, we'll pick $k_3 = 2$. Then

$$
k_3 = 2, \quad k_1 = 1, \quad k_2 = -5.
$$

This implies that according to our calculations

$$
x^{(1)} - 5x^{(2)} + 2x^{(3)} = 0,
$$

which you can easily check. Thus the three vectors are linearly dependent.

Example: Let

$$
x^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
$$

In order to test the linear dependence of these vectors, we have to rowreduce the matrix

$$
\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \overset{E_{12}(-2)}{\longrightarrow} \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}
$$

$$
\overset{M_2(-1/3)}{\longrightarrow} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}
$$

$$
\overset{E_{21}(-2)}{\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

This gives rise to the system

$$
\begin{cases} k_1 & = & 0 \\ k_2 & = & 0 \end{cases},
$$

and thus the vectors are linearly independent.

Example: Let's finish with a minor bang: consider the following five vectors:

$$x^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \quad x^{(3)} = \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad x^{(4)} = \begin{pmatrix} -3 \\ 0 \\ -1 \\ 3 \end{pmatrix}, \quad x^{(5)} = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}.$$

We have to rowreduce the following matrix:

$$\begin{aligned}
\begin{pmatrix} 1 & -1 & -2 & -3 & 1 \\ 2 & 0 & -1 & 0 & 1 \\ 2 & 3 & 1 & -1 & -3 \\ 3 & 1 & 0 & 3 & 1 \end{pmatrix} &\overset{E_{12}(-2),\ E_{13}(-2),\ E_{14}(-3)}{\longrightarrow} \begin{pmatrix} 1 & -1 & -2 & -3 & 1 \\ 0 & 2 & 3 & 6 & -1 \\ 0 & 5 & 5 & 5 & -5 \\ 0 & 4 & 6 & 12 & -2 \end{pmatrix} \\
&\overset{M_3(1/5)}{\longrightarrow} \begin{pmatrix} 1 & -1 & -2 & -3 & 1 \\ 0 & 2 & 3 & 6 & -1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 4 & 6 & 12 & -2 \end{pmatrix} \\
&\overset{P_{23}}{\longrightarrow} \begin{pmatrix} 1 & -1 & -2 & -3 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 2 & 3 & 6 & -1 \\ 0 & 4 & 6 & 12 & -2 \end{pmatrix} \\
&\overset{E_{21}(1),\ E_{23}(-2),\ E_{24}(-4)}{\longrightarrow} \begin{pmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 8 & 2 \end{pmatrix} \\
&\overset{E_{31}(1),\ E_{32}(-1),\ E_{34}(-2)}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

We obtain the system

$$\left\{ \begin{array}{rcl} k_1 + 2k_4 + k_5 & = & 0 \\ k_2 - 3k_4 - 2k_5 & = & 0 \\ k_3 + 4k_4 + k_5 & = & 0 \end{array} \right. .$$

We get to choose both k_4 and k_5 at will. This implies that there's more than one relationship between the five vectors. Let's choose $k_4 = 1$, $k_5 = 0$. Then

$$k_1 = -2, \quad k_2 = 3, \quad k_3 = -4,$$

and

$$-2x^{(1)} + 3x^{(2)} - 4x^{(3)} + x^{(4)} = 0,$$

which is easily verified. Alternatively, choose $k_4 = 0$ and $k_5 = 1$. Then

$$k_1 = -1, \quad k_2 = 2, \quad k_3 = -1,$$

given rise to

$$-x^{(1)} + 2x^{(2)} - x^{(3)} + x^{(45)} = 0,$$

which is also easily verified.

Let's finish with a few remarks.

- Suppose we are working in dimension N, meaning all vectors are that size. Suppose M vectors are given. If $M > N$ then these vectors are always linearly dependent: it is impossible to have more independent vectors than the dimension. In terms of the linear system we have to solve, this means that we have more variables than equations. Of course, we'll get to choose some of the variables to be non-zero. Therefore the vectors are linearly dependent.

- If we have fewer or an equal number of vectors than their dimension, than *typically* these vectors are linearly independent. Of course, if things go wrong and the numbers conspire against us, it is possible for the vectors to be dependent. This happens quite frequently in course notes, homework problems and exams.

Lecture 19. Systems and linear algebra IV: the inverse matrix and determinants

Note: For this lecture, the matrix A we'll consider has to be square, so it's number of rows equals its number of columns.

The inverse matrix

Let's look back at solving linear systems

$$Ax = b.$$

Now, if these were just numbers, we'd have

$$x = \frac{b}{A}.$$

For matrices, this doesn't make any sense: what does it mean to divide by a matrix? Today we'll construct a matrix which we'll denote by A^{-1} for which

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

We call this matrix the inverse of A. If these hold, then

$$\begin{aligned} & & Ax &= b \\ &\Rightarrow & A^{-1}Ax &= A^{-1}b \\ &\Rightarrow & Ix &= A^{-1}b \\ &\Rightarrow & x &= A^{-1}b, \end{aligned}$$

and we have a solution for our system! Several questions have to be answered:

- Is this a "good" way to solve $Ax = b$? Here "good" means efficient: if there's a faster way to solve the system, then why bother? We'll answer this question in a little while.
- Is this always possible? The answer to this is clearly "no"! We already know that this only works for square matrices. We'll see below that it doesn't even work for all square matrices. Bummer!
- How do we find A^{-1}? We'll answer this question now. It'll also provide us with an answer to the first question.

Here's the method for finding an inverse matrix. We use its definition: the inverse matrix is the matrix X that satisfies the equation

$$AX = I.$$

If we can find an X that is the unique solution to this equation, then $A^{-1} = X$. The way to do this is to rowreduce the augmented matrix $(A|I) \to (I|X)$, in which case $A^{-1} = X$. Let's do an example.

Example: Let

$$A = \begin{pmatrix} 1 & 1 & -10 \\ -1 & 0 & 10 \\ 1 & 4 & -5 \end{pmatrix}.$$

The above says that we have to solve the system $AX = I$, thus we consider the augmented matrix $(A|I)$ and rowreduce it.

$$\begin{aligned}
\begin{pmatrix} 1 & 1 & -10 & | & 1 & 0 & 0 \\ -1 & 0 & 10 & | & 0 & 1 & 0 \\ 1 & 4 & -5 & | & 0 & 0 & 1 \end{pmatrix} &\overset{E_{12}(1),\ E_{13}(-1)}{\longrightarrow} \begin{pmatrix} 1 & 1 & -10 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 1 & 1 & 0 \\ 0 & 3 & 5 & | & -1 & 0 & 1 \end{pmatrix} \\
&\overset{E_{21}(-1),\ E_{23}(-3)}{\longrightarrow} \begin{pmatrix} 1 & 0 & -10 & | & 0 & -1 & 0 \\ 0 & 1 & 0 & | & 1 & 1 & 0 \\ 0 & 0 & 5 & | & -4 & -3 & 1 \end{pmatrix} \\
&\overset{M_3(1/5)}{\longrightarrow} \begin{pmatrix} 1 & 0 & -10 & | & 0 & -1 & 0 \\ 0 & 1 & 0 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & -4/5 & -3/5 & 1/5 \end{pmatrix} \\
&\overset{E_{31}(10)}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & | & -8 & -7 & 2 \\ 0 & 1 & 0 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & -4/5 & -3/5 & 1/5 \end{pmatrix},
\end{aligned}$$

from which it follows that

$$X = \begin{pmatrix} -8 & -7 & 2 \\ 1 & 1 & 0 \\ -4/5 & -3/5 & 1/5 \end{pmatrix}.$$

Since this is the only solution we can find, it follows that $A^{-1} = X$, thus

$$A^{-1} = \begin{pmatrix} -8 & -7 & 2 \\ 1 & 1 & 0 \\ -4/5 & -3/5 & 1/5 \end{pmatrix}.$$

Since this is the first inverse matrix we've found, our confidence level may be somewhat below par. Let's check that this is right (confidence level low or not, this is always a good idea!):

$$A^{-1}A = \begin{pmatrix} 1 & 1 & -10 \\ -1 & 0 & 10 \\ 1 & 4 & -5 \end{pmatrix} \begin{pmatrix} -8 & -7 & 2 \\ 1 & 1 & 0 \\ -4/5 & -3/5 & 1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the same result for AA^{-1}, so that we have indeed found the inverse matrix.

At this point we can answer the question of whether solving a system of equations by finding an inverse matrix is an efficient way of doing things. The answer is negative:

to construct the inverse matrix we have to solve in effect as many linear systems as the dimension of the matrix: as opposed to augmenting the matrix with one column, we have to augment it with as many columns as the size of the matrix. Thus the rowreduction is more work. However, if we'd have to solve the same system over and over again, just with a different right-hand side (believe me, this happens), this would be beneficial: once we have the inverse matrix, all we have to do is multiply this inverse matrix with the new right-hand side and we're done!

Example: Let's see. Is calculating the inverse always this straightforward? You know when I'm asking the answer is "no". Let's do another example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

our favorite matrix! We have to rowreduce $(A|I)$ again:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right) \xrightarrow{E_{12}(-4),\ E_{13}(-7)} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{M_2(-1/3)} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4/3 & -1/3 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{E_{21}(-2),\ E_{23}(6)} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -5/3 & 2/3 & 0 \\ 0 & 1 & 2 & 4/3 & -1/3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right).$$

Since the RREF of the matrix A is not I, we have to conclude that it is impossible to rowreduce $(A|I)$ to the form $(I|X)$, so that the inverse of A does not exist. In this case, the matrix A is called **singular**.

Let's recap all of this: when we're presented with a square matrix A, and we want to find its inverse, we construct the augmented matrix $(A|I)$. We rowreduce this matrix. If the result of this is an augmented matrix of the form $(I|X)$, then A is **non-singular** and it has an inverse, and $A^{-1} = X$. Otherwise, A is called **singular**, and it does not have an inverse.

Determinants

For any square matrix A, there is a number, $\det(A)$, called the **determinant** of A, which dtermines if the matrix is singular or nonsingular. If the determinant is zero, the matrix is singular. Otherwise it is nonsingular.

How do we calculate a determinant? This is not an easy question. I'll give you the general method below, but I won't show you why it works. That would require more time than we can afford at this point. The theory of determinants is a large and important part of linear algebra. Unfortunately, we can only do a little bit of it.

Let's see how things work for matrices of different sizes:

- 1×1 **matrices:** in this case $\det(A) = A$, the only entry in the matrix.
- 2×2 **matrices:** let

  $$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

  Then

  $$\det(A) = \alpha\delta - \beta\gamma.$$

  In the previous lecture we've seen that two two-dimensional vectors are linearly independent if the RREF of the matrix formed from them has a zero row, or in other words if this matrix is singular. In that case one of the vectors is a multiple of the other. Thus

  $$v^{(1)} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix},$$

  where c is the proportionality constant. Then

  $$A = (v^{(1)} \;\; v^{(2)}) = \begin{pmatrix} v_1 & cv_1 \\ v_2 & cv_2 \end{pmatrix}.$$

  The determinant of this matrix is $\det(A) = v_1 cv_2 - cv_1 v_2 = 0$, showing that the matrix A is indeed singular.

  Now, let $f(x)$ and $g(x)$ be two functions. We form the two vectors

  $$v^{(1)} = \begin{pmatrix} f \\ f' \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} g \\ g' \end{pmatrix}.$$

  These two vectors are linearly dependent if the two functions f and g are multiples of each other, thus if the functions are linearly dependent. The determinant of the matrix $A = (v^{(1)} \;\; v^{(2)})$ is

  $$\det \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = fg' - f'g = W(f, g).$$

  Thus the Wronskian of two functions can be written as a determinant. This is good to know: now we have a pretty good idea how we'll generalize the idea of more than two functions being linearly dependent: we'll form the determinant of the matrix with as columns the functions and their derivatives.
- 3×3 **matrices:** let

  $$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

  Then

  $$\det(A) = (aei + bfg + cdh) - (gec + hfa + idb).$$

  There's many good ways to remember this formula. The easiest one might be to construct the auxialiary matrix

  $$\begin{pmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{pmatrix}.$$

Now the determinant of A is formed by adding the terms constructed of multiplying the entries with entries one over and down, and subtracting the terms constructed of multiplying the entries with entries one up and over.

- **For any size matrix:** transform A to its $RREF$. In order to do this you'll need several $E_{ij}(k)$'s, several $M_j(\alpha)$'s and some P_{ij}'s. Once you're done, the determinant of A can be found as:

$$\det(A) = \frac{(-1)^{\text{number of } P_{ij}\text{'s}}}{\text{product of all the } \alpha\text{'s}} \times \begin{cases} 1 & \text{if } RREF = I \\ 0 & \text{if } RREF \neq I \end{cases}.$$

We won't prove this result, but it provides an efficient way of computing the determinant of a matrix of size greater than three.

Example: In the earlier part of this lecture we calculated the inverse matrix of

$$A = \begin{pmatrix} 1 & 1 & -10 \\ -1 & 0 & 10 \\ 1 & 4 & -5 \end{pmatrix}.$$

We now calculate the determinant of this matrix two different ways:

1. Using the definition of the determinant for a three-by-three matrix, we have

$$\begin{aligned} \det(A) &= 1 * 0 * (-5) + 1 * 1 * 10 + (-1) * 4 * (-10) \\ &\quad - 1 * 0 * (-10) - 4 * 10 * 1 - (-1) * 1 * (-5) \\ &= 0 + 10 + 40 - 0 - 40 - 5 \\ &= 5. \end{aligned}$$

2. Using the transformation to RREF, we get

$$\det(A) = \frac{1}{1/5} \times 1 = 5.$$

Of course we obtain the same result both ways.

Other properties of determinants that we will not prove are: (i) $\det(AB) = \det(A)\det(B)$, and (ii) $\det(A^T) = \det(A)$.

In summary, the following facts are all equivalent:

- $\det(A) = 0$,
- A is singular,
- the columns of A are linearly dependent,
- the rows of A are linearly dependent,
- A^{-1} does not exist
- $RREF(A)$ has zero rows
- The system $Ax = b$ either has no solutions, or an infinite number of solutions.

Lecture 20. Systems and linear algebra V: eigenvalues and eigenvectors

We start with a note on solving linear systems. Suppose we want to solve

$$Ax = b,$$

with A a square matrix. Suppose we know that the determinant of A is non-zero. Then A is nonsingular and it has an inverse. Thus

$$x = A^{-1}b,$$

which is the only solution of the system.

In particular, if A is nonsingular, then $x = 0$ is the only solution of the system

$$Ax = 0.$$

Thus, if we want interesting solutions of the equation $Ax = 0$, we want $\det(A) = 0$, so that A is singular.

When we multiply a matrix by a vector, we get a vector. This new vector typically is very different from the original vector: it will have a different length, and a different direction. Given a square matrix A, are there particular vectors ξ for which the new vector $A\xi$ is in the same direction as the original vector ξ? That means that the new vector $A\xi$ is just a scalar multiple of the old vector ξ. Using equations:

$$A\xi = \lambda\xi,$$

where λ is the scaling factor. Such vectors are called **eigenvectors**, and the corresponding scaling factors are called **eigenvalues**[1].

Let's rewrite this equation:

$$\begin{aligned} & A\xi = \lambda\xi \\ \Rightarrow \quad & A\xi - \lambda\xi = 0 \\ \Rightarrow \quad & A\xi - \lambda I\xi = 0 \\ \Rightarrow \quad & (A - \lambda I)\xi = 0. \end{aligned}$$

Now, we don't want any zero eigenvectors: that's not interesting: the zero vector *always* gets mapped to the zero vector. In fact, in the homeworks and exams that you'll do on this topic, if you *ever* write down a zero eigenvector, you will be moved to the back of the class instantaneouly! **Don't ever write down a zero eigenvector!** Okay, how can we get a non-zero eigenvector: according to our starting note, the equation $(A - \lambda I)\xi = 0$ will only have the zero solution, unless we impose that the determinant of $A - \lambda I$ is zero. So, we better impose this:

$$\boxed{\det(A - \lambda I) = 0.}$$

[1]The word "eigen" is German. It means "self": the eigenvectors of a matrix are the vectors that get mapped by the matrix to themselves, up to a scaling factor, the eigenvalue.

This equation is called the **characteristic equation** of the matrix A: it determines the eigenvalues since the eigenvectors don't enter in to this equation. Let's do a few examples.

Example: Consider

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

The characteristic equation is

$$\begin{aligned}
& \det\left[\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0 \\
\Rightarrow \quad & \det\left[\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = 0 \\
\Rightarrow \quad & \det\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = 0 \\
\Rightarrow \quad & (-2-\lambda)^2 - 1 = 0 \\
\Rightarrow \quad & (\lambda+2)^2 = 1 \\
\Rightarrow \quad & \lambda + 2 = \pm 1 \\
\Rightarrow \quad & \lambda_1 = -1, \ \lambda_2 = -3.
\end{aligned}$$

Example: Consider

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

The characteristic equation is

$$\begin{aligned}
& \det\left[\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = 0 \\
\Rightarrow \quad & \det\begin{pmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{pmatrix} = 0 \\
\Rightarrow \quad & (1-\lambda)(5-\lambda)(9-\lambda) + 84 + 96 \\
& -21(5-\lambda) - 8(9-\lambda) - 48(1-\lambda) = 0 \\
\Rightarrow \quad & -\lambda^3 + 15\lambda^2 + 10\lambda = 0. \\
\Rightarrow \quad & \lambda = 0; \ -\lambda^2 + 15\lambda + 10 = 0 \\
\Rightarrow \quad & \lambda_1 = 0; \ \lambda_{2,3} = \frac{-15 \pm \sqrt{225+40}}{-2} \\
\Rightarrow \quad & \lambda_1 = 0; \ \lambda_{2,3} = \frac{15 \mp \sqrt{265}}{2}.
\end{aligned}$$

We see that to construct the characteristic equation, we just subtract λ from the elements on the diagonal of the matrix, then we equate the determinant of the resulting matrix to zero. We also see that the characteristic equation of an $n \times n$ matrix will be an n-th degree polynomial. Thus, in general, an $n \times n$ matrix will have n eigenvalues, including multiplicities. Keep in mind that these eigenvalues can be complex numbers!

Example: Let

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

The eigenvalues of A are determined by

$$\begin{aligned} & & \det\begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} &= 0 \\ &\Rightarrow & (1-\lambda)(3-\lambda)+1 &= 0 \\ &\Rightarrow & \lambda^2 - 4\lambda + 4 &= 0 \\ &\Rightarrow & (\lambda-2)^2 &= 0 \\ &\Rightarrow & \lambda_1 = \lambda_2 &= 2. \end{aligned}$$

Thus the matrix A has two eigenvalues, both equal to 2.

Example: Let

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

The eigenvalues of A are determined by

$$\begin{aligned} & & \det\begin{pmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{pmatrix} &= 0 \\ &\Rightarrow & (2-\lambda)^2+1 &= 0 \\ &\Rightarrow & (\lambda-2)^2 &= -1 \\ &\Rightarrow & \lambda_{1,2} - 2 &= \pm i \\ &\Rightarrow & \lambda_{1,2} &= 2 \pm i \\ &\Rightarrow & \lambda_1 = 2+i;\ \lambda_2 &= 2-i. \end{aligned}$$

Thus the matrix A has two complex conjugate eigenvalues.

How about the eigenvectors? This question is now easier to answer: we know how to find the eigenvalues. For any eigenvalue λ we found, all we have to do is find the *nontrivial* (*i.e.*, non-zero) solution ξ of

$$(A - \lambda I)\xi = 0.$$

By virtue of the eigenvalues making $A - \lambda I$ a singular matrix, we know this equation has nontrivial solutions for ξ. This calculation needs to be done for all eigenvalues, if we want to find all eigenvectors of A. Let's do some examples.

Example: Let

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

We've seen above that the eigenvalues of this matr-ix are $\lambda_1 = -1$ and $\lambda_2 = -3$. Let's find their corresponding eigenvectors.

- **$\lambda_1 = -1$:** we have to find the nontrivial solution of
$$\begin{pmatrix} -2-\lambda_1 & 1 \\ 1 & -2-\lambda_1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
The most systematic way to do this is to row-reduce the resulting matrix:
$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \stackrel{P_{12}}{\longrightarrow} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \stackrel{E_{12}(1)}{\longrightarrow} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$
Thus
$$\xi_1 - \xi_2 = 0,$$
and we can choose $\xi_1 = 1$, which gives $\xi_2 = 1$. The eigenvector corresponding to $\lambda_1 = -1$ is
$$\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
or any multiple thereof. You should verify that indeed
$$A\xi = -\xi.$$

- **$\lambda_2 = -3$:** We now have to repeat the above for $\lambda_2 = -3$. Now we have to find the nontrivial solution of
$$\begin{pmatrix} -2-\lambda_2 & 1 \\ 1 & -2-\lambda_2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
The most systematic (which is not the fastest) way to do this is to row-reduce the resulting matrix:
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \stackrel{E_{12}(-1)}{\longrightarrow} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$
Thus
$$\xi_1 + \xi_2 = 0,$$
and we can choose $\xi_1 = 1$, which gives $\xi_2 = -1$. The eigenvector corresponding to $\lambda_1 = -1$ is
$$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
or any multiple thereof. You should verify that
$$A\xi = -3\xi.$$

Example: Let
$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$
You should check that the eigenvalues of this matrix are $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = -1$. Hence -1 is an eigenvalue of multiplicity 2.

- $\lambda_1 = 2$**:** we have to rowreduce the matrix $A - 2I$:

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \overset{P_{13}}{\longrightarrow} \begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix}$$

$$\overset{E_{12}(-1),\ E_{13}(2)}{\longrightarrow} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}$$

$$\overset{M_2(-1/3)}{\longrightarrow} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{pmatrix}$$

$$\overset{E_{21}(-1),\ E_{23}(-3)}{\longrightarrow} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

  Thus our eigenvector can be chosen to be

$$\xi = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

  or any scalar multiple thereof.

- $\lambda_2 = \lambda_3 = -1$**:** we have to rowreduce the matrix $A + I$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \overset{E_{12}(-1),\ E_{13}(-1)}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

  Thus

$$\xi_1 + \xi_2 + \xi_3 = 0.$$

  The eigenvectors are of the form

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} -\xi_2 - \xi_3 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \xi_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \xi_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

  It follows from this that there are two linearly independent eigenvectors. We can choose

$$\xi^{(1)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

  As before you should check that for both of these vectors $A\xi = -\xi$.

In the previous example, we found that an eigenvalue of multiplicity two had two linearly independent eigenvectors. You might be tempted to conclude that corresponding to a certain eigenvalue, there are always as many linearly independent eigenvectors as the multiplicity of that eigenvalue. This is not the case, as we'll see in the next example.

Example: Let

$$A = -\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

You can (and you should) check that the two eigenvalues of this matrix are equal. They are $\lambda_1 = \lambda_2 = 2$. However, when we rowreduce $A - 2I$ we get

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \overset{P_{12}}{\longrightarrow} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \overset{E_{12}(1)}{\longrightarrow} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

so that there's only one eigenvector, namely

$$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In the previous example, there was only one eigenvector for an eigenvalue of multiplicity two. The general rule is that there are **at most** as many eigenvectors as the multiplicity of the eigenvalue. Further, there is always at least one eigenvector for any eigenvalue.

Example: Let's do a final example: what happens when the eigenvalues are complex? Well, the answer is pretty straightforward: the corresponding eigenvectors will be complex as well. Consider

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

We've seen that $\lambda_1 = 2 + 1$, and $\lambda_2 = 2 - 1$. What are the eigenvectors? For the first one, we have to rowreduce $A - (2 + i)I$:

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \overset{P_{12}}{\longrightarrow} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$
$$\overset{E_{12}(i)}{\longrightarrow} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix},$$

so that the eigenvector can be chosen to be

$$\xi^{(1)} = \begin{pmatrix} i \\ 1 \end{pmatrix},$$

similarly,

$$\xi^{(2)} = \begin{pmatrix} -i \\ 1 \end{pmatrix},$$

corresponding to $\lambda_2 = 2 - i$. Note that the second eigenvector is the complex conjugate of the first one. This is also a general rule: for real matrices, eigenvectors corresponding to complex conjugate eigenvalues can be chosen to be complex conjugate.

Lecture 21. First-order linear systems

General considerations

Consider a system of N linear differential equations. In its most general form, this is written as

$$\left\{\begin{array}{rcl} x_1' & = & p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1N}(t)x_N + g_1(t) \\ x_2' & = & p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2N}(t)x_N + g_2(t) \\ & \vdots & \\ x_N' & = & p_{N1}(t)x_1 + p_{N2}(t)x_2 + \ldots + p_{NN}(t)x_N + g_N(t) \end{array}\right. .$$

With what we've learned about matrices, we can write this as

$$\boxed{x' = Px + g,}$$

where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \qquad g = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix},$$

and

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{pmatrix}..$$

Example: Consider the third-order equation

$$y''' + 2y' + 5y = 7.$$

Since this equation is of third order, we introduce three variables:

$$\left\{\begin{array}{rcl} y_1 & = & y, \\ y_2 & = & y', \\ y_3 & = & y'', \end{array}\right.$$

Then

$$\left\{\begin{array}{rcl} y_1' & = & y' = y_2, \\ y_2' & = & y'' = y_3, \\ y_3' & = & y''' = -2y' - 5y + 7 = -2y_2 - 5y_1 + 7. \end{array}\right.$$

Thus our corresponding first-order system is

$$\left\{\begin{array}{rcl} y_1' & = & y_2, \\ y_2' & = & y_3, \\ y_3' & = & -2y_2 - 5y_1 + 7. \end{array}\right.$$