Students' Solutions Manual

for

Applied Linear Algebra

by

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To the Student

These solutions are a resource for students studying the second edition of our text *Applied Linear Algebra*, published by Springer in 2018. An expanded solutions manual is available for registered instructors of courses adopting it as the textbook.

Using the Manual

The material taught in this book requires an active engagement with the exercises, and we urge you not to read the solutions in advance. Rather, you should use the ones in this manual as a means of verifying that your solution is correct. (It is our hope that all solutions appearing here are correct; errors should be reported to the authors.) If you get stuck on an exercise, try skimming the solution to get a hint for how to proceed, but then work out the exercise yourself. The more you can do on your own, the more you will learn. Please note: for students taking a course based on *Applied Linear Algebra*, copying solutions from this Manual can place you in violation of academic honesty. In particular, many solutions here just provide the final answer, and for full credit one must also supply an explanation of how this is found.

Acknowledgements

We thank a number of people, who are named in the text, for corrections to the solutions manuals that accompanied the first edition. Thanks to Alexander Voronov and his students Jacob Boerjan and Cassandra Chanthamontry for further corrections to the current manual. Of course, as authors, we take full responsibility for all errors that may yet appear. We encourage readers to inform us of any misprints, errors, and unclear explanations that they may find, and will accordingly update this manual on a timely basis. Corrections will be posted on the text's dedicated web site:

`http://www.math.umn.edu/~olver/ala2.html`

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Students' Solutions Manual for
Chapter 1: Linear Algebraic Systems

1.1.1. (*b*) Reduce the system to $6u + v = 5$, $-\frac{5}{2}v = \frac{5}{2}$; then use Back Substitution to solve for $u = 1$, $v = -1$.

(*d*) Reduce the system to $2u - v + 2w = 2$, $-\frac{3}{2}v + 4w = 2$, $-w = 0$; then solve for $u = \frac{1}{3}$, $v = -\frac{4}{3}$, $w = 0$.

(*f*) Reduce the system to $x + z - 2w = -3$, $-y + 3w = 1$, $-4z - 16w = -4$, $6w = 6$; then solve for $x = 2$, $y = 2$, $z = -3$, $w = 1$.

♡ 1.1.3. (a) With Forward Substitution, we just start with the top equation and work down. Thus $2x = -6$ so $x = -3$. Plugging this into the second equation gives $12 + 3y = 3$, and so $y = -3$. Plugging the values of x and y in the third equation yields $-3 + 4(-3) - z = 7$, and so $z = -22$.

1.2.1. (a) 3×4, (*b*) 7, (*c*) 6, (*d*) $(\,-2 \quad 0 \quad 1 \quad 2\,)$, (*e*) $\begin{pmatrix} 0 \\ 2 \\ -6 \end{pmatrix}$.

1.2.2. Examples: (a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, (*c*) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 3 \end{pmatrix}$, (*e*) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

1.2.4. (*b*) $A = \begin{pmatrix} 6 & 1 \\ 3 & -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} u \\ v \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$;

(*d*) $A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & -1 & 3 \\ 3 & 0 & -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$;

(*f*) $A = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 2 & -1 & 2 & -1 \\ 0 & -6 & -4 & 2 \\ 1 & 3 & 2 & -1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ -5 \\ 2 \\ 1 \end{pmatrix}$.

1.2.5. (*b*) $u + w = -1$, $u + v = -1$, $v + w = 2$. The solution is $u = -2$, $v = 1$, $w = 1$.

(*c*) $3x_1 - x_3 = 1$, $-2x_1 - x_2 = 0$, $x_1 + x_2 - 3x_3 = 1$.

The solution is $x_1 = \frac{1}{5}$, $x_2 = -\frac{2}{5}$, $x_3 = -\frac{2}{5}$.

1.2.6. (*a*) $\mathrm{I} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, $\mathrm{O} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(*b*) $\mathrm{I} + \mathrm{O} = \mathrm{I}$, $\mathrm{I}\,\mathrm{O} = \mathrm{O}\,\mathrm{I} = \mathrm{O}$. No, it does not.

1.2.7. (*b*) undefined, (*c*) $\begin{pmatrix} 3 & 6 & 0 \\ -1 & 4 & 2 \end{pmatrix}$, (*f*) $\begin{pmatrix} 1 & 11 & 9 \\ 3 & -12 & -12 \\ 7 & 8 & 8 \end{pmatrix}$,

1.2.9. $1, 6, 11, 16$.

1.2.10. (*a*) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

1.2.11. (*a*) True.

1.2.14. $B = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$ where x, y are arbitrary.

1.2.17. $A\,\mathrm{O}_{n\times p} = \mathrm{O}_{m\times p}$, $\mathrm{O}_{l\times m}\,A = \mathrm{O}_{l\times n}$.

1.2.19. False: for example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

1.2.22. (*a*) A must be a square matrix. (*b*) By associativity, $A\,A^2 = A\,A\,A = A^2\,A = A^3$.

1.2.25. (*a*) $X = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$.

♢ 1.2.29. (*a*) The i^{th} entry of $A\,\mathbf{z}$ is $1\,a_{i1} + 1\,a_{i2} + \cdots + 1\,a_{in} = a_{i1} + \cdots + a_{in}$, which is the i^{th} row sum. (*b*) Each row of W has $n-1$ entries equal to $\dfrac{1}{n}$ and one entry equal to $\dfrac{1-n}{n}$ and so its row sums are $(n-1)\,\dfrac{1}{n} + \dfrac{1-n}{n} = 0$. Therefore, by part (a), $W\,\mathbf{z} = \mathbf{0}$. Consequently, the row sums of $B = AW$ are the entries of $B\,\mathbf{z} = AW\,\mathbf{z} = A\,\mathbf{0} = \mathbf{0}$, and the result follows.

♡ 1.2.34. (*a*) This follows by direct computation. (*b*) (*i*)

$$\begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 3 & -6 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 5 & -6 \end{pmatrix}.$$

1.3.1. (*b*) $u = 1,\ v = -1$; (*d*) $x_1 = \frac{11}{3},\ x_2 = -\frac{10}{3},\ x_3 = -\frac{2}{3}$;
(*f*) $a = \frac{1}{3},\ b = 0,\ c = \frac{4}{3},\ d = -\frac{2}{3}$.

1.3.2. (*a*) $\left(\begin{array}{cc|c} 1 & 7 & 4 \\ -2 & -9 & 2 \end{array}\right) \xrightarrow{2R_1+R_2} \left(\begin{array}{cc|c} 1 & 7 & 4 \\ 0 & 5 & 10 \end{array}\right)$. Back Substitution yields $x_2 = 2,\ x_1 = -10$.

(*c*) $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right) \xrightarrow{4R_1+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right) \xrightarrow{\frac{3}{2}R_2+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right)$.

Back Substitution yields $z = 3,\ y = 16,\ x = 29$.

(e) $\left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & -3 & 2 & 0 & 0 \\ -4 & 0 & 0 & 7 & -5 \end{array}\right)$ reduces to $\left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 2 & -3 & 6 \\ 0 & 0 & 0 & -5 & 15 \end{array}\right)$.

Back Substitution yields $x_4 = -3,\ x_3 = -\frac{3}{2},\ x_2 = -1,\ x_1 = -4$.

1.3.3. (a) $3x + 2y = 2,\ -4x - 3y = -1$; solution: $x = 4,\ y = -5$.

1.3.4. (a) Regular: $\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 \\ 0 & \frac{7}{2} \end{pmatrix}$.

(d) Not regular: $\begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ 3 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 5 \\ 0 & 5 & -7 \end{pmatrix}$.

1.3.5. (a) $\left(\begin{array}{cc|c} -\mathrm{i} & 1+\mathrm{i} & -1 \\ 1-\mathrm{i} & 1 & -3\,\mathrm{i} \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -\mathrm{i} & 1+\mathrm{i} & -1 \\ 0 & 1-2\,\mathrm{i} & 1-2\,\mathrm{i} \end{array}\right)$;

use Back Substitution to obtain the solution $y = 1,\ x = 1 - 2\,\mathrm{i}$.

♢ 1.3.11. (a) Set $l_{ij} = \begin{cases} a_{ij}, & i > j, \\ 0, & i \le j, \end{cases}$ $u_{ij} = \begin{cases} a_{ij}, & i < j, \\ 0, & i \ge j, \end{cases}$ $d_{ij} = \begin{cases} a_{ij}, & i = j, \\ 0, & i \ne j. \end{cases}$

(b) $L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $U = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

1.3.15. (a) Add -2 times the second row to the first row of a $2 \times n$ matrix.
(c) Add -5 times the third row to the second row of a $3 \times n$ matrix.

1.3.16. (a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

1.3.20. (a) Upper triangular; (b) both upper and lower unitriangular; (d) lower unitriangular.

1.3.21. (a) $L = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}$, (c) $L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$,

(e) $L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$, $U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

1.3.25. (a) $a_{ij} = 0$ for all $i \ne j$; (c) $a_{ij} = 0$ for all $i > j$ and $a_{ii} = 1$ for all i.

1.3.27. False. For instance $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is regular. Only if the zero appear in the $(1,1)$ position does it automatically preclude regularity of the matrix.

1.3.32. (*a*) $\mathbf{x} = \begin{pmatrix} -1 \\ \frac{2}{3} \end{pmatrix}$, (*c*) $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, (*e*) $\mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ \frac{5}{2} \end{pmatrix}$.

1.3.33. (*a*) $L = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$, $U = \begin{pmatrix} -1 & 3 \\ 0 & 11 \end{pmatrix}$; $\mathbf{x}_1 = \begin{pmatrix} -\frac{5}{11} \\ \frac{2}{11} \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} \frac{9}{11} \\ \frac{3}{11} \end{pmatrix}$.

(*c*) $L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{2}{9} & \frac{5}{3} & 1 \end{pmatrix}$, $U = \begin{pmatrix} 9 & -2 & -1 \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$; $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} -2 \\ -9 \\ -1 \end{pmatrix}$.

(*e*) $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & \frac{3}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & -1 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & -\frac{7}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 4 \end{pmatrix}$; $\mathbf{x}_1 = \begin{pmatrix} \frac{5}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} \frac{1}{14} \\ -\frac{5}{14} \\ \frac{1}{14} \\ \frac{1}{2} \end{pmatrix}$.

1.4.1. (*a*) Nonsingular, (*c*) nonsingular, (*e*) singular.

1.4.2. (*a*) Regular and nonsingular, (*c*) nonsingular.

1.4.5. (*b*) $x_1 = 0$, $x_2 = -1$, $x_3 = 2$; (*d*) $x = -\frac{13}{2}$, $y = -\frac{9}{2}$, $z = -1$, $w = -3$.

1.4.6. True. All regular matrices are nonsingular.

1.4.11. (*a*) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, (*c*) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

1.4.13. (*b*) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

1.4.14. (*a*) True, since interchanging the same pair of rows twice brings you back to where you started.

1.4.16. (*a*) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. (*b*) True.

1.4.19. Let $\mathbf{r}_i, \mathbf{r}_j$ denote the rows of the matrix in question. After the first elementary row operation, the rows are $\mathbf{r}_i$ and $\mathbf{r}_j + \mathbf{r}_i$. After the second, they are $\mathbf{r}_i - (\mathbf{r}_j + \mathbf{r}_i) = -\mathbf{r}_j$ and $\mathbf{r}_j + \mathbf{r}_i$. After the third operation, we are left with $-\mathbf{r}_j$ and $\mathbf{r}_j + \mathbf{r}_i + (-\mathbf{r}_j) = \mathbf{r}_i$.

1.4.21. (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix};$

(c) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & -3 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 9 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix};$

(e) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 4 & -1 & 2 \\ 7 & -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ 7 & -29 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 3 \end{pmatrix}.$

1.4.22. (b) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -3 \\ 1 & 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & \frac{5}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & -3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix};$

solution: $x = 4, \quad y = 0, \quad z = 1, \quad w = 1.$

1.4.26. False. Changing the permuation matrix typically changes the pivots.

1.5.1. (b) $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$

1.5.3. (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

1.5.6. (a) $A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.$

(b) $C = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}, \quad C^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{pmatrix}.$

1.5.9. (a) $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

1.5.13. Since c is a scalar, $\left(\frac{1}{c}A^{-1}\right)(cA) = \frac{1}{c}\,c\,A^{-1}A = \mathrm{I}.$

1.5.16. If all the diagonal entries are nonzero, then $D^{-1}D = \mathrm{I}$. On the other hand, if one of diagonal entries is zero, then all the entries in that row are zero, and so D is singular.

♢ 1.5.19. (a) $A = \mathrm{I}^{-1}A\,\mathrm{I}$. (b) If $B = S^{-1}AS$, then $A = S\,B\,S^{-1} = T^{-1}B\,T$, where $T = S^{-1}$.

1.5.21. (a) $B\,A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) $A\,X = \mathrm{I}$ does not have a solution. Indeed, the first column of this matrix equation is the linear system $\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which has no solutions since $x - y = 1$, $y = 0$, and $x + y = 0$ are incompatible.

1.5.25. (b) $\begin{pmatrix} -\frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & -\frac{1}{8} \end{pmatrix}$; (d) no inverse; (f) $\begin{pmatrix} -\frac{5}{8} & \frac{1}{8} & \frac{5}{8} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{7}{8} & -\frac{3}{8} & \frac{1}{8} \end{pmatrix}$.

1.5.26. (b) $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -8 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$; (d) not possible;

(f) $$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 5 \\ 2 & 1 & 2 \end{pmatrix}.$$

1.5.28. (a) $\begin{pmatrix} -\frac{\mathrm{i}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\mathrm{i}}{2} \end{pmatrix}$, (c) $\begin{pmatrix} \mathrm{i} & 0 & -1 \\ 1-\mathrm{i} & -\mathrm{i} & 1 \\ -1 & -1 & -\mathrm{i} \end{pmatrix}$.

1.5.31. (b) $\begin{pmatrix} \frac{5}{17} & \frac{2}{17} \\ -\frac{1}{17} & \frac{3}{17} \end{pmatrix}\begin{pmatrix} 2 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, (d) $\begin{pmatrix} 9 & -15 & -8 \\ 6 & -10 & -5 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$,

(f) $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 2 & -1 & -1 & 0 \\ 2 & -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 11 \\ -7 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \\ -2 \end{pmatrix}$.

1.5.32. (b) $\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$; (d) singular matrix; (f) $\begin{pmatrix} \frac{1}{8} \\ -\frac{1}{2} \\ \frac{5}{8} \end{pmatrix}$.

1.5.33. (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 4 \\ -7 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -7 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{7} \\ 0 & 1 \end{pmatrix}$,

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 5 \\ 1 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & \frac{7}{3} \\ 0 & 0 & 1 \end{pmatrix}$,

(f) $\begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & -4 & 1 & 5 \\ 1 & 2 & -1 & -1 \\ 3 & 1 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 3 & -\frac{4}{3} & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

1.5.34. (b) $\begin{pmatrix} -8 \\ 3 \end{pmatrix}$, (d) $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, (f) $\begin{pmatrix} \frac{7}{3} \\ 2 \\ 5 \\ -\frac{5}{3} \end{pmatrix}$.

1.6.1. (b) $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 2 \end{pmatrix}$, (f) $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$.

1.6.2. $A^T = \begin{pmatrix} 3 & 1 \\ -1 & 2 \\ -1 & 1 \end{pmatrix}$, $B^T = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 0 & 4 \end{pmatrix}$,

$(AB)^T = B^T A^T = \begin{pmatrix} -2 & 0 \\ 2 & 6 \end{pmatrix}$, $(BA)^T = A^T B^T = \begin{pmatrix} -1 & 6 & -5 \\ 5 & -2 & 11 \\ 3 & -2 & 7 \end{pmatrix}$.

1.6.5. $(ABC)^T = C^T B^T A^T$

1.6.8. (a) $(AB)^{-T} = ((AB)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1} = A^{-T} B^{-T}$.

(b) $AB = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, so $(AB)^{-T} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$,

while $A^{-T} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, $B^{-T} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, so $A^{-T} B^{-T} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.

♢ 1.6.13. (a) Using Exercise 1.6.12, $a_{ij} = \mathbf{e}_i^T A \mathbf{e}_j = \mathbf{e}_i^T B \mathbf{e}_j = b_{ij}$ for all i, j.

(b) Two examples: $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

1.6.17. (b) $a = -1, \ b = 2, \ c = 3$.

♢ 1.6.20. True. Invert both sides of the equation $A^T = A$, and use Lemma 1.32.

1.6.25. (a) $\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 3 & 2 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$.

♢ 1.6.30. Write $A = L D V$, then $A^T = V^T D U^T = \widetilde{L}\,\widetilde{U}$, where $\widetilde{L} = V^T$ and $\widetilde{U} = D\,\widetilde{L}$. Thus, A^T is regular since the diagonal entries of $\widetilde{U}$, which are the pivots of A^T, are the same as those of D and U, which are the pivots of A.

1.7.1. (*b*) The solution is $x = -4,\ y = -5,\ z = -1$. Gaussian Elimination and Back Substitution requires 17 multiplications and 11 additions; Gauss–Jordan uses 20 multiplications and 11 additions; computing $A^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ 2 & -8 & -5 \\ \frac{3}{2} & -5 & -3 \end{pmatrix}$ takes 27 multiplications and 12 additions, while multiplying $A^{-1}\mathbf{b} = \mathbf{x}$ takes another 9 multiplications and 6 additions.

♢ 1.7.4. We begin by proving (1.63). We must show that $1 + 2 + 3 + \ldots + (n-1) = \frac{1}{2}\,n(n-1)$ for $n = 2, 3, \ldots$. For $n = 2$ both sides equal 1. Assume that (1.63) is true for $n = k$. Then $1+2+3+\ldots+(k-1)+k = \frac{1}{2}\,k(k-1)+k = \frac{1}{2}\,k(k+1)$, so (1.63) is true for $n = k+1$. Now the first equation in (1.64) follows if we note that $1 + 2 + 3 + \ldots + (n-1) + n = \frac{1}{2}\,n(n+1)$.

1.7.9. (*a*) $\begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}.$

1.7.11. For $n = 4$: (*a*)

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{pmatrix},$$

(*b*) $(\,2, 3, 3, 2\,)^T$. (*c*) $x_i = i(n-i+1)/2$ for $i = 1, \ldots, n$.

♠ 1.7.13. $$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{5} & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} \\ 0 & 0 & \frac{18}{5} \end{pmatrix}$$

1.7.16. (*a*) $\begin{pmatrix} -8 \\ 4 \end{pmatrix}$, (*b*) $\begin{pmatrix} -10 \\ -4.1 \end{pmatrix}$, (*c*) $\begin{pmatrix} -8.1 \\ -4.1 \end{pmatrix}$. (*d*) Partial pivoting reduces the effect of round off errors and results in a significantly more accurate answer.

1.7.20. (*a*) $\begin{pmatrix} \frac{6}{5} \\ -\frac{13}{5} \\ -\frac{9}{5} \end{pmatrix} = \begin{pmatrix} 1.2 \\ -2.6 \\ -1.8 \end{pmatrix}$, (*c*) $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

1.7.21. (a) $\begin{pmatrix} -\frac{1}{13} \\ \frac{8}{13} \end{pmatrix} = \begin{pmatrix} -.0769 \\ .6154 \end{pmatrix}$, (c) $\begin{pmatrix} \frac{2}{121} \\ \frac{38}{121} \\ \frac{59}{242} \\ -\frac{56}{121} \end{pmatrix} = \begin{pmatrix} .0165 \\ .3141 \\ .2438 \\ -.4628 \end{pmatrix}$.

1.8.1. (a) Unique solution: $(-\frac{1}{2}, -\frac{3}{4})^T$; (c) no solutions;
(e) infinitely many solutions: $(5 - 2z, 1, z, 0)^T$, where z is arbitrary.

1.8.2. (b) Incompatible; (c) $(1, 0)^T$;
(d) $(1 + 3x_2 - 2x_3, x_2, x_3)^T$, where x_2 and x_3 are arbitrary.

1.8.5. (a) $\left(1 + \mathrm{i} - \frac{1}{2}(1 + \mathrm{i})y, y, -\mathrm{i}\right)^T$, where y is arbitrary.

1.8.7. (b) 1, (d) 3, (f) 1.

1.8.8. (b) $\begin{pmatrix} 2 & 1 & 3 \\ -2 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$,

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}$,

(f) $(\,0 \quad -1 \quad 2 \quad 5\,) = (\,1\,)(\,0 \quad -1 \quad 2 \quad 5\,)$.

1.8.11. Example: (a) $x^2 + y^2 = 1$, $x^2 - y^2 = 2$.

1.8.13. True. For example, take a matrix in row echelon form with r pivots, e.g., the matrix A with $a_{ii} = 1$ for $i = 1, \ldots, r$, and all other entries equal to 0.

1.8.17. 1.

♢ 1.8.21. By Proposition 1.39, A can be reduced to row echelon form U by a sequence of elementary row operations. Therefore, as in the proof of the LU decomposition, $A = E_1 E_2 \cdots E_N U$ where $E_1^{-1}, \ldots, E_N^{-1}$ are the elementary matrices representing the row operations. If A is singular, then $U = Z$ must have at least one all-zero row.

1.8.22. (a) $x = z$, $y = z$, where z is arbitrary; (c) $x = y = z = 0$.

1.8.23. (a) $\left(\frac{1}{3}y, y\right)^T$, where y is arbitrary;
(c) $\left(-\frac{11}{5}z + \frac{3}{5}w, \frac{2}{5}z - \frac{6}{5}w, z, w\right)^T$, where z and w are arbitrary.

1.8.27. (b) $k = 0$ or $k = \frac{1}{2}$.

1.9.1. (a) Regular matrix, reduces to upper triangular form $U = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$, so its determinant is 2.

(b) Singular matrix, row echelon form $U = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$, so its determinant is 0.

(d) Nonsingular matrix, reduces to upper triangular form $U = \begin{pmatrix} -2 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ after one row interchange, so its determinant is 6.

1.9.4. (a) True. By Theorem 1.52, A is nonsingular, so, by Theorem 1.18, A^{-1} exists.

(c) False. For $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have

$$\det(A+B) = \det \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} = 0 \neq -1 = \det A + \det B.$$

1.9.5. By (1.85, 86) and commutativity of numeric multiplication,

$$\det B = \det(S^{-1} A S) = \det S^{-1} \det A \det S = \frac{1}{\det S} \det A \, \det S = \det A.$$

♢ 1.9.9.

$$\det \begin{pmatrix} a & b \\ c+ka & d+kb \end{pmatrix} = ad + akb - bc - bka = ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - ad = -(ad - bc) = -\det \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\det \begin{pmatrix} ka & kb \\ c & d \end{pmatrix} = kad - kbc = k(ad - bc) = k \det \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = ad - b0 = ad.$$

1.9.13.

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} =$$

$$\begin{aligned} & a_{11} a_{22} a_{33} a_{44} - a_{11} a_{22} a_{34} a_{43} - a_{11} a_{23} a_{32} a_{44} + a_{11} a_{23} a_{34} a_{42} - a_{11} a_{24} a_{33} a_{42} \\ & + a_{11} a_{24} a_{32} a_{43} - a_{12} a_{21} a_{33} a_{44} + a_{12} a_{21} a_{34} a_{43} + a_{12} a_{23} a_{31} a_{44} - a_{12} a_{23} a_{34} a_{41} \\ & + a_{12} a_{24} a_{33} a_{41} - a_{12} a_{24} a_{31} a_{43} + a_{13} a_{21} a_{32} a_{44} - a_{13} a_{21} a_{34} a_{42} - a_{13} a_{22} a_{31} a_{44} \\ & + a_{13} a_{22} a_{34} a_{41} - a_{13} a_{24} a_{32} a_{41} + a_{13} a_{24} a_{31} a_{42} - a_{14} a_{21} a_{32} a_{43} + a_{14} a_{21} a_{33} a_{42} \\ & + a_{14} a_{22} a_{31} a_{43} - a_{14} a_{22} a_{33} a_{41} + a_{14} a_{23} a_{32} a_{41} - a_{14} a_{23} a_{31} a_{42}. \end{aligned}$$

♡ 1.9.20. (a) By direct substitution:

$$ax + by = a \frac{pd - bq}{ad - bc} + b \frac{aq - pc}{ad - bc} = p, \qquad cx + dy = c \frac{pd - bq}{ad - bc} + d \frac{aq - pc}{ad - bc} = q.$$

(b) (i) $x = -\frac{1}{10} \det \begin{pmatrix} 13 & 3 \\ 0 & 2 \end{pmatrix} = -2.6, \quad y = -\frac{1}{10} \det \begin{pmatrix} 1 & 13 \\ 4 & 0 \end{pmatrix} = 5.2.$

Students' Solutions Manual for
Chapter 2: Vector Spaces and Bases

2.1.1. *Commutativity of Addition*:
$$(x + \mathrm{i}\,y) + (u + \mathrm{i}\,v) = (x + u) + \mathrm{i}\,(y + v) = (u + \mathrm{i}\,v) + (x + \mathrm{i}\,y).$$
Associativity of Addition:
$$\begin{aligned}(x + \mathrm{i}\,y) + \big[\,(u + \mathrm{i}\,v) + (p + \mathrm{i}\,q)\,\big] &= (x + \mathrm{i}\,y) + \big[\,(u + p) + \mathrm{i}\,(v + q)\,\big] \\ &= (x + u + p) + \mathrm{i}\,(y + v + q) \\ &= \big[\,(x + u) + \mathrm{i}\,(y + v)\,\big] + (p + \mathrm{i}\,q) = \big[\,(x + \mathrm{i}\,y) + (u + \mathrm{i}\,v)\,\big] + (p + \mathrm{i}\,q).\end{aligned}$$
Additive Identity: $\mathbf{0} = 0 = 0 + \mathrm{i}\,0$ and
$$(x + \mathrm{i}\,y) + 0 = x + \mathrm{i}\,y = 0 + (x + \mathrm{i}\,y).$$
Additive Inverse: $-(x + \mathrm{i}\,y) = (-x) + \mathrm{i}\,(-y)$ and
$$(x + \mathrm{i}\,y) + \big[\,(-x) + \mathrm{i}\,(-y)\,\big] = 0 = \big[\,(-x) + \mathrm{i}\,(-y)\,\big] + (x + \mathrm{i}\,y).$$
Distributivity:
$$\begin{aligned}(c + d)\,(x + \mathrm{i}\,y) &= (c + d)\,x + \mathrm{i}\,(c + d)\,y = (c\,x + d\,x) + \mathrm{i}\,(c\,y + d\,y) \\ &= c\,(x + \mathrm{i}\,y) + d\,(x + \mathrm{i}\,y), \\ c[\,(x + \mathrm{i}\,y) + (u + \mathrm{i}\,v)\,] &= c\,(x + u) + (y + v) = (c\,x + c\,u) + \mathrm{i}\,(c\,y + c\,v) \\ &= c\,(x + \mathrm{i}\,y) + c\,(u + \mathrm{i}\,v).\end{aligned}$$
Associativity of Scalar Multiplication:
$$c\,[\,d\,(x + \mathrm{i}\,y)\,] = c\,[\,(d\,x) + \mathrm{i}\,(d\,y)\,] = (c\,d\,x) + \mathrm{i}\,(c\,d\,y) = (c\,d)\,(x + \mathrm{i}\,y).$$
Unit for Scalar Multiplication: $1\,(x + \mathrm{i}\,y) = (1\,x) + \mathrm{i}\,(1\,y) = x + \mathrm{i}\,y$.

Note: Identifying the complex number $x + \mathrm{i}\,y$ with the vector $(\,x, y\,)^T \in \mathbb{R}^2$ respects the operations of vector addition and scalar multiplication, and so we are in effect reproving that $\mathbb{R}^2$ is a vector space.

2.1.4. (a) $(\,1, 1, 1, 1\,)^T$, $(\,1, -1, 1, -1\,)^T$, $(\,1, 1, 1, 1\,)^T$, $(\,1, -1, 1, -1\,)^T$. (b) Obviously not.

2.1.6. (a) $f(x) = -\,4\,x + 3$.

2.1.7. (a) $\begin{pmatrix} x - y \\ x\,y \end{pmatrix}$, $\begin{pmatrix} e^x \\ \cos y \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, which is a constant function.

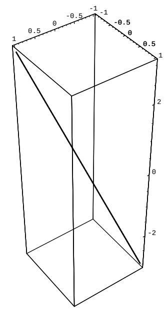
(b) Their sum is $\begin{pmatrix} x - y + e^x + 1 \\ x\,y + \cos y + 3 \end{pmatrix}$. Multiplied by $-\,5$ is $\begin{pmatrix} -\,5\,x + 5\,y - 5\,e^x - 5 \\ -\,5\,x\,y - 5\cos y - 15 \end{pmatrix}$.

(c) The zero element is the constant function $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

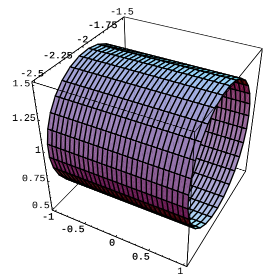
2.1.11. (j) Let $\mathbf{z} = c\,\mathbf{0}$. Then $\mathbf{z} + \mathbf{z} = c\,(\mathbf{0} + \mathbf{0}) = c\,\mathbf{0} = \mathbf{z}$, and so, as in the proof of (h), $\mathbf{z} = \mathbf{0}$.

2.2.2. (*a*) Not a subspace; (*c*) subspace; (*e*) not a subspace; (*g*) subspace.

2.2.3. (*a*) Subspace:



(*e*) Not a subspace:



2.2.7. (*b*) Not a subspace; (*d*) subspace.

2.2.10. (*a*) Not a subspace; (*c*) not a subspace; (*f*) subspace.

2.2.14. (*a*) Vector space; (*c*) vector space;
(*e*) not a vector space: If f is non-negative, then $-1\,f = -f$ is not (unless $f \equiv 0$).

2.2.15. (*b*) Subspace; (*d*) subspace, (*e*) not a subspace.

2.2.16. (*a*) Subspace; (*c*) not a subspace: the zero function does not satisfy the condition;
(*e*) subspace; (*g*) subspace.

2.2.20. $\nabla \cdot (c\,\mathbf{v} + d\,\mathbf{w}) = c\,\nabla \cdot \mathbf{v} + d\,\nabla \cdot \mathbf{w} = 0$ whenever $\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{w} = 0$ and $c, d, \in \mathbb{R}$.

♢ 2.2.22. (*a*) If $\mathbf{v}, \mathbf{w} \in W \cap Z$, then $\mathbf{v}, \mathbf{w} \in W$, so $c\,\mathbf{v} + d\,\mathbf{w} \in W$ because W is a subspace, and $\mathbf{v}, \mathbf{w} \in Z$, so $\mathbf{c}\,\mathbf{v} + d\,\mathbf{w} \in Z$ because Z is a subspace, hence $c\,\mathbf{v} + d\,\mathbf{w} \in W \cap Z$.

♡ 2.2.24. (*b*) Since the only common solution to $x = y$ and $x = 3y$ is $x = y = 0$, the lines only intersect at the origin. Moreover, every $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} 3\,b \\ b \end{pmatrix}$, where $a = -\frac{1}{2}\,x + \frac{3}{2}\,y$, $b = \frac{1}{2}\,x - \frac{1}{2}\,y$, can be written as a sum of vectors on each line.

♢ 2.2.30. (*a*) By induction, we can show that, for $n \geq 1$ and $x > 0$,

$$f^{(n)}(x) = \frac{Q_{n-1}(x)}{x^{2\,n}}\, e^{-1/x^2},$$

where $Q_{n-1}(x)$ is a polynomial of degree $n-1$. Thus,

$$\lim_{x \to 0^+} f^{(n)}(x) = \lim_{x \to 0^+} \frac{Q_{n-1}(x)}{x^{2\,n}}\, e^{-1/x^2} = Q_{n-1}(0) \lim_{y \to \infty} y^{2\,n}\, e^{-y} = 0 = \lim_{x \to 0^-} f^{(n)}(x),$$

because the exponential e^{-y} goes to zero faster than any power of y goes to ∞.

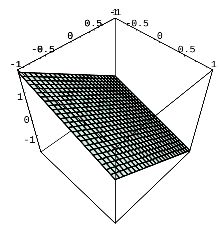
2.3.1. $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}.$

2.3.3. (*a*) Yes, since $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$;

(*c*) No, since the vector equation $\begin{pmatrix} 3 \\ 0 \\ -1 \\ -2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -1 \\ 3 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ does not have a solution.

2.3.4. (*a*) No, (*c*) yes, (*e*) yes.

2.3.5. (*b*) The plane $z = -\frac{3}{5}x - \frac{6}{5}y$:



2.3.8. (*a*) They span $\mathcal{P}^{(2)}$ since $ax^2+bx+c = \frac{1}{2}(a-2b+c)(x^2+1)+\frac{1}{2}(a-c)(x^2-1)+b(x^2+x+1)$.

2.3.9. (*a*) Yes, (*c*) no, (*e*) no.

2.3.10. (*a*) $\sin 3x = \cos\left(3x - \frac{1}{2}\pi\right)$, (*c*) $3\cos 2x + 4\sin 2x = 5\cos\left(2x - \tan^{-1}\frac{4}{3}\right)$.

2.3.13. (*a*) e^{2x}; (*c*) e^{3x}, 1; (*e*) $e^{-x/2}\cos\frac{\sqrt{3}}{2}x$, $e^{-x/2}\sin\frac{\sqrt{3}}{2}x$.

2.3.15. (*a*) $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\,\mathbf{f}_1(x) + \mathbf{f}_2(x) - \mathbf{f}_3(x)$; (*b*) not in the span;

(*c*) $\begin{pmatrix} 1-2x \\ -1-x \end{pmatrix} = \mathbf{f}_1(x) - \mathbf{f}_2(x) - \mathbf{f}_3(x)$.

♢ 2.3.19. (*a*) If $\mathbf{v} = \sum_{j=1}^{m} c_j\,\mathbf{v}_j$ and $\mathbf{v}_j = \sum_{i=1}^{n} a_{ij}\,\mathbf{w}_i$, then $\mathbf{v} = \sum_{i=1}^{n} b_i\,\mathbf{v}_i$ where $b_i = \sum_{j=1}^{m} a_{ij}\,c_j$, or, in vector language, $\mathbf{b} = A\,\mathbf{c}$.

(*b*) Every $\mathbf{v} \in V$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and hence, by part (*a*), a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_m$, which shows that $\mathbf{w}_1, \ldots, \mathbf{w}_m$ also span V.

2.3.21. (*b*) Linearly dependent; (*d*) linearly independent; (*f*) linearly dependent.

2.3.24. (*a*) Linearly dependent; (*c*) linearly independent; (*e*) linearly dependent.

2.3.27. Yes, when it is the zero vector.

♢ 2.3.31. (*a*) Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent,

$$\mathbf{0} = c_1\,\mathbf{v}_1 + \cdots + c_k\,\mathbf{v}_k = c_1\,\mathbf{v}_1 + \cdots + c_k\,\mathbf{v}_k + 0\,\mathbf{v}_{k+1} + \cdots + 0\,\mathbf{v}_n$$

if and only if $c_1 = \cdots = c_k = 0$.

(*b*) This is false. For example, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, are linearly dependent, but the subset consisting of just $\mathbf{v}_1$ is linearly independent.

2.3.32. (*a*) They are linearly dependent since $(x^2-3)+2(2-x)-(x-1)^2 \equiv 0$.
(*b*) They do not span $\mathcal{P}^{(2)}$.

2.3.33. (*b*) Linearly independent; (*d*) linearly independent; (*f*) linearly dependent.

♡ 2.3.37. (*a*) If $c_1 f_1(x) + \cdots + c_n f_n(x) \equiv 0$, then $c_1 f_1(x_i) + \cdots + c_n f_n(x_i) = 0$ at all sample points, and so $c_1 \mathbf{f}_1 + \cdots + c_n \mathbf{f}_n = \mathbf{0}$. Thus, linear dependence of the functions implies linear dependence of their sample vectors.
(*b*) Sampling $f_1(x) = 1$ and $f_2(x) = x^2$ at $x_1 = -1$ and $x_2 = 1$ produces the linearly dependent sample vectors $\mathbf{f}_1 = \mathbf{f}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

2.4.1. (*a*) Basis; (*b*) not a basis (*d*) not a basis.

2.4.2. (*a*) Not a basis; (*c*) not a basis.

2.4.5. (*a*) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$; (*b*) $\begin{pmatrix} \frac{3}{4} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} \\ 0 \\ 1 \end{pmatrix}$.

♡ 2.4.7. (*a*) (*i*) Left-handed basis; (*iii*) not a basis.

2.4.8. (*b*) The condition $p(1) = 0$ says $a + b + c = 0$, so

$$p(x) = (-b-c)x^2 + bx + c = b(-x^2+x) + c(-x^2+1).$$

Therefore $-x^2+x,\ -x^2+1$ is a basis, and so $\dim = 2$.

2.4.10. (*a*) $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$, $\dim = 1$; (*b*) $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$, $\dim = 2$.

2.4.13. (*a*) The sample vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ are linearly independent and hence form a basis for $\mathbb{R}^4$ — the space of sample functions.

(*b*) Sampling x produces $\begin{pmatrix} 0 \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2+\sqrt{2}}{8}\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix} - \frac{2-\sqrt{2}}{8}\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$.

2.4.17. (*a*) $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; dimension $= 3$.

◇ 2.4.20. (*a*) $m \le n$ as otherwise $\mathbf{v}_1, \ldots, \mathbf{v}_m$ would be linearly dependent. If $m = n$ then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent and hence, by Theorem 2.31 span all of $\mathbb{R}^n$. Since every vector in their span also belongs to V, we must have $V = \mathbb{R}^n$.
(*c*) (*i*) Example: $\left(1, 1, \frac{1}{2}\right)^T, (1,0,0)^T, (0,1,0)^T$.

◇ 2.4.21. (a) By Theorem 2.31, we only need prove linear independence. If $\mathbf{0} = c_1 A\mathbf{v}_1 + \cdots + c_n A\mathbf{v}_n = A(c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n)$, then, since A is nonsingular, $c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n = \mathbf{0}$, and hence $c_1 = \cdots = c_n = 0$.

(b) $A\mathbf{e}_i$ is the i^{th} column of A, and so a basis consists of the column vectors of the matrix.

◇ 2.4.24. For instance, take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_3$. In fact, there are infinitely many different ways of writing this vector as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

2.5.1. (a) Image: all $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ such that $\frac{3}{4} b_1 + b_2 = 0$; kernel spanned by $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$.

(c) Image: all $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ such that $-2b_1 + b_2 + b_3 = 0$; kernel spanned by $\begin{pmatrix} -\frac{5}{4} \\ -\frac{7}{8} \\ 1 \end{pmatrix}$.

2.5.2. (a) $\begin{pmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$: plane; (b) $\begin{pmatrix} \frac{1}{4} \\ \frac{3}{8} \\ 1 \end{pmatrix}$: line; (e) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$: point.

2.5.4. (a) $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$; (b) the general solution is $\mathbf{x} = \begin{pmatrix} 1+t \\ 2+t \\ 3+t \end{pmatrix}$, where t is arbitrary.

2.5.7. In each case, the solution is $\mathbf{x} = \mathbf{x}^\star + \mathbf{z}$, where $\mathbf{x}^\star$ is the particular solution and $\mathbf{z}$ belongs to the kernel: (b) $\mathbf{x}^\star = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{z} = z\begin{pmatrix} -\frac{2}{7} \\ -\frac{1}{7} \\ 1 \end{pmatrix}$; (d) $\mathbf{x}^\star = \begin{pmatrix} \frac{5}{6} \\ 1 \\ -\frac{2}{3} \end{pmatrix}$, $\mathbf{z} = \mathbf{0}$.

2.5.8. The kernel has dimension $n-1$, with basis $-r^{k-1}\mathbf{e}_1 + \mathbf{e}_k = \left(-r^{k-1}, 0, \ldots, 0, 1, 0, \ldots, 0\right)^T$ for $k = 2, \ldots n$. The image has dimension 1, with basis $(1, r^n, r^{2n} \ldots, r^{(n-1)n})^T$.

2.5.12. $\mathbf{x}_1^\star = \begin{pmatrix} -2 \\ \frac{3}{2} \end{pmatrix}$, $\mathbf{x}_2^\star = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}$; $\mathbf{x} = \mathbf{x}_1^\star + 4\mathbf{x}_2^\star = \begin{pmatrix} -6 \\ \frac{7}{2} \end{pmatrix}$.

2.5.14. (a) By direct matrix multiplication: $A\mathbf{x}_1^\star = A\mathbf{x}_2^\star = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$.

(b) The general solution is $\mathbf{x} = \mathbf{x}_1^\star + t(\mathbf{x}_2^\star - \mathbf{x}_1^\star) = (1-t)\mathbf{x}_1^\star + t\mathbf{x}_2^\star = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$.

2.5.19. False: in general, $(A+B)\mathbf{x}^\star = (A+B)\mathbf{x}_1^\star + (A+B)\mathbf{x}_2^\star = \mathbf{c} + \mathbf{d} + B\mathbf{x}_1^\star + A\mathbf{x}_2^\star$, and the third and fourth terms don't necessarily add up to $\mathbf{0}$.

$\diamondsuit$ 2.5.20. $\operatorname{img} A = \mathbb{R}^n$, and so A must be a nonsingular matrix.

2.5.21. (*a*) image: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; coimage: $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$; kernel: $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$; cokernel: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

(*c*) image: $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$; coimage: $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -3 \\ 2 \end{pmatrix}$; kernel: $\begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$; cokernel: $\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$.

2.5.23. (*i*) rank $= 1$; $\dim \operatorname{img} A = \dim \operatorname{coimg} A = 1$, $\dim \ker A = \dim \operatorname{coker} A = 1$;
kernel basis: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$; cokernel basis: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; compatibility conditions: $2 b_1 + b_2 = 0$;
example: $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

(*iii*) rank $= 2$; $\dim \operatorname{img} A = \dim \operatorname{coimg} A = 2$, $\dim \ker A = 0$, $\dim \operatorname{coker} A = 1$;
kernel: $\{\mathbf{0}\}$; cokernel basis: $\begin{pmatrix} -\frac{20}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}$; compatibility conditions: $-\frac{20}{13} b_1 + \frac{3}{13} b_2 + b_3 = 0$;
example: $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(*v*) rank $= 2$; $\dim \operatorname{img} A = \dim \operatorname{coimg} A = 2$, $\dim \ker A = 1$, $\dim \operatorname{coker} A = 2$; kernel basis: $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$; cokernel basis: $\begin{pmatrix} -\frac{9}{4} \\ \frac{1}{4} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 0 \\ 1 \end{pmatrix}$; compatibility: $-\frac{9}{4} b_1 + \frac{1}{4} b_2 + b_3 = 0$,
$\frac{1}{4} b_1 - \frac{1}{4} b_2 + b_4 = 0$; example: $\mathbf{b} = \begin{pmatrix} 2 \\ 6 \\ 3 \\ 1 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

2.5.24. (*b*) $\dim = 1$; basis: $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$; (*d*) $\dim = 3$; basis: $\begin{pmatrix} 1 \\ 0 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -8 \\ 7 \end{pmatrix}$.

2.5.26. (*b*) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$.

2.5.29. Both sets are linearly independent and hence span a three-dimensional subspace of $\mathbb{R}^4$. Moreover, $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3$, $\mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ all lie in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and hence, by Theorem 2.31(*d*) also form a basis for the subspace.

2.5.35. We know $\operatorname{img} A \subset \mathbb{R}^m$ is a subspace of dimension $r = \operatorname{rank} A$. In particular, $\operatorname{img} A = \mathbb{R}^m$ if and only if it has dimension $m = \operatorname{rank} A$.

♢ 2.5.38. If $\mathbf{v} \in \ker A$ then $A\mathbf{v} = \mathbf{0}$ and so $B A \mathbf{v} = B\mathbf{0} = \mathbf{0}$, so $\mathbf{v} \in \ker(B A)$. The first statement follows from setting $B = A$.

2.5.41. True. If $\ker A = \ker B \subset \mathbb{R}^n$, then both matrices have n columns, and so $n - \operatorname{rank} A = \dim \ker A = \dim \ker B = n - \operatorname{rank} B$.

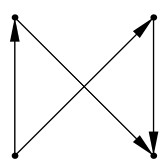
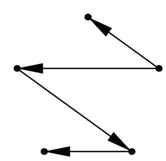
♢ 2.5.44. Since we know $\dim \operatorname{img} A = r$, it suffices to prove that $\mathbf{w}_1, \ldots, \mathbf{w}_r$ are linearly independent. Given

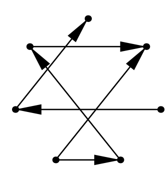
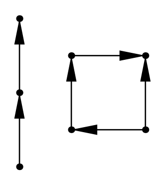
$$\mathbf{0} = c_1 \mathbf{w}_1 + \cdots + c_r \mathbf{w}_r = c_1 A \mathbf{v}_1 + \cdots + c_r A \mathbf{v}_r = A(c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r),$$

we deduce that $c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r \in \ker A$, and hence can be written as a linear combination of the kernel basis vectors:

$$c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r = c_{r+1} \mathbf{v}_{r+1} + \cdots + c_n \mathbf{v}_n.$$

But $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, and so $c_1 = \cdots = c_r = c_{r+1} = \cdots = c_n = 0$, which proves linear independence of $\mathbf{w}_1, \ldots, \mathbf{w}_r$.

2.6.2. (a)  (c) 

(e)  or, equivalently, 

2.6.3. (a) $\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$; (c) $\begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$.

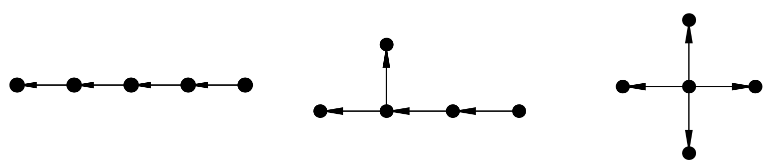
2.6.4. (a) 1 circuit: $\begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$; (c) 2 circuits: $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

♡ 2.6.7. (a) Tetrahedron: $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

number of circuits $= \dim \operatorname{coker} A = 3$, number of faces $= 4$.

♡ 2.6.9. (a) (i) $\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$, (iii) $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$.

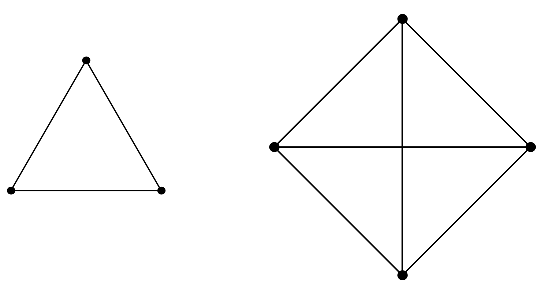
(b)



$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

♡ 2.6.10. G_3 G_4

(a)



(b) $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$.

♡ 2.6.14. (a) Note that P permutes the rows of A, and corresponds to a relabeling of the vertices of the digraph, while Q permutes its columns, and so corresponds to a relabeling of the edges. (b) (i) Equivalent, (iii) inequivalent, (v) equivalent.

Students' Solutions Manual for

Chapter 3: Inner Products and Norms

3.1.2. (b) No — not positive definite; (d) no — not bilinear; (f) yes.

3.1.4. (b) Bilinearity:

$$\begin{aligned}
\langle c\,\mathbf{u}+d\,\mathbf{v}\,,\mathbf{w}\,\rangle &= 4\,(cu_1+d\,v_1)\,w_1+2\,(c\,u_1+d\,v_1)\,w_2+2\,(c\,u_2+d\,v_2)\,w_1+\\
&\qquad +4\,(cu_2+d\,v_2)\,w_2+(c\,u_3+d\,v_3)\,w_3\\
&= c\,(4\,u_1\,w_1+2\,u_1\,w_2+2\,u_2\,w_1+4\,u_2\,w_2+u_3\,w_3)+\\
&\qquad +d\,(4\,v_1\,w_1+2\,v_1\,w_2+2\,v_2\,w_1+4\,v_2\,w_2+v_3\,w_3)\\
&= c\,\langle\,\mathbf{u}\,,\mathbf{w}\,\rangle+d\,\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle,\\
\langle\,\mathbf{u}\,,c\,\mathbf{v}+d\,\mathbf{w}\,\rangle &= 4\,u_1\,(cv_1+d\,w_1)+2\,u_1\,(cv_2+d\,w_2)+2\,u_2\,(cv_1+d\,w_1)+\\
&\qquad +4\,u_2\,(cv_2+d\,w_2)+u_3\,(cv_3+d\,w_3)\\
&= c\,(4\,u_1\,v_1+2\,u_1\,v_2+2\,u_2\,v_1+4\,u_2\,v_2+u_3\,v_3)+\\
&\qquad +d\,(4\,u_1\,w_1+2\,u_1\,w_2+2\,u_2\,w_1+4\,u_2\,w_2+u_3\,w_3)\\
&= c\,\langle\,\mathbf{u}\,,\mathbf{v}\,\rangle+d\,\langle\,\mathbf{u}\,,\mathbf{w}\,\rangle.
\end{aligned}$$

Symmetry:

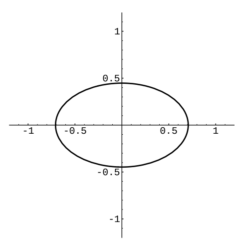
$$\begin{aligned}
\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle &= 4\,v_1\,w_1+2\,v_1\,w_2+2\,v_2\,w_1+4\,v_2\,w_2+v_3\,w_3\\
&= 4\,w_1\,v_1+2\,w_1\,v_2+2\,w_2\,v_1+4\,w_2\,v_2+w_3\,v_3=\langle\,\mathbf{w}\,,\mathbf{v}\,\rangle.
\end{aligned}$$

Positivity:

$$\langle\,\mathbf{v}\,,\mathbf{v}\,\rangle = 4\,v_1^2+4\,v_1\,v_2+4\,v_2^2+v_3^2=(2\,v_1+v_2)^2+3\,v_2^2+v_3^2>0$$
$$\text{for all}\quad \mathbf{v}=(\,v_1,v_2,v_3\,)^T\neq\mathbf{0},$$

because it is a sum of non-negative terms, at least one of which is strictly positive.

3.1.5. (b) Ellipse with semi-axes $\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{5}}$:



◇ 3.1.7. $\|\,c\,\mathbf{v}\,\| = \sqrt{\langle\,c\,\mathbf{v}\,,c\,\mathbf{v}\,\rangle}=\sqrt{c^2\,\langle\,\mathbf{v}\,,\mathbf{v}\,\rangle}=|\,c\,|\;\|\,\mathbf{v}\,\|.$

◇ 3.1.10. (a) Choosing $\mathbf{v}=\mathbf{x}$, we have $0=\langle\,\mathbf{x}\,,\mathbf{x}\,\rangle=\|\,\mathbf{x}\,\|^2$, and hence $\mathbf{x}=\mathbf{0}$.

(b) Rewrite the condition as $0=\langle\,\mathbf{x}\,,\mathbf{v}\,\rangle-\langle\,\mathbf{y}\,,\mathbf{v}\,\rangle=\langle\,\mathbf{x}-\mathbf{y}\,,\mathbf{v}\,\rangle$ for all $\mathbf{v}\in V$. Now use part (a) to conclude that $\mathbf{x}-\mathbf{y}=\mathbf{0}$ and so $\mathbf{x}=\mathbf{y}$.

♢ 3.1.12. (a) $\| \mathbf{u}+\mathbf{v} \|^2 - \| \mathbf{u}-\mathbf{v} \|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle - \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle$

$$= \big(\langle \mathbf{u}, \mathbf{u} \rangle + 2 \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \big) - \big(\langle \mathbf{u}, \mathbf{u} \rangle - 2 \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \big) = 4 \langle \mathbf{u}, \mathbf{v} \rangle.$$

3.1.15. Using (3.2), $\mathbf{v} \cdot (A\mathbf{w}) = \mathbf{v}^T A \mathbf{w} = (A^T \mathbf{v})^T \mathbf{w} = (A^T \mathbf{v}) \cdot \mathbf{w}$.

3.1.21. (a) $\langle 1, x \rangle = \frac{1}{2}$, $\| 1 \| = 1$, $\| x \| = \frac{1}{\sqrt{3}}$;

(c) $\langle x, e^x \rangle = 1$, $\| x \| = \frac{1}{\sqrt{3}}$, $\| e^x \| = \sqrt{\tfrac{1}{2}(e^2 - 1)}$.

3.1.22. (b) $\langle f, g \rangle = 0$, $\| f \| = \sqrt{\tfrac{2}{3}}$, $\| g \| = \sqrt{\tfrac{56}{15}}$.

3.1.23. (a) Yes; (b) no, since it fails positivity: for instance, $\int_{-1}^{1} (1-x)^2 x \, dx = -\frac{4}{3}$.

3.1.26. No. For example, on $[-1, 1]$, $\| 1 \| = \sqrt{2}$, but $\| 1 \|^2 = 2 \neq \| 1^2 \| = \sqrt{2}$.

♢ 3.1.30. (a) To prove the first bilinearity condition:

$$\langle c f + d g, h \rangle = \int_a^b \big[c f(x) + d g(x) \big] h(x) w(x) \, dx$$
$$= c \int_a^b f(x) h(x) w(x) \, dx + d \int_a^b g(x) h(x) w(x) \, dx = c \langle f, h \rangle + d \langle g, h \rangle.$$

The second has a similar proof, or follows from symmetry, cf. Exercise 3.1.9.
To prove symmetry:

$$\langle f, g \rangle = \int_a^b f(x) g(x) w(x) \, dx = \int_a^b g(x) f(x) w(x) \, dx = \langle g, f \rangle.$$

As for positivity, $\langle f, f \rangle = \int_a^b f(x)^2 w(x) \, dx \geq 0$. Moreover, since $w(x) > 0$ and the integrand is continuous, Exercise 3.1.29 implies that $\langle f, f \rangle = 0$ if and only if $f(x)^2 w(x) \equiv 0$ for all x, and so $f(x) \equiv 0$.

(b) If $w(x_0) < 0$, then, by continuity, $w(x) < 0$ for $x_0 - \delta \leq x \leq x_0 + \delta$ for some $\delta > 0$. Now choose $f(x) \not\equiv 0$ so that $f(x) = 0$ whenever $| x - x_0 | > \delta$. Then

$$\langle f, f \rangle = \int_a^b f(x)^2 w(x) \, dx = \int_{x_0-\delta}^{x_0+\delta} f(x)^2 w(x) \, dx < 0, \quad \text{violating positivity.}$$

3.1.32. (a) $\langle f, g \rangle = \frac{2}{3}$, $\| f \| = 1$, $\| g \| = \sqrt{\tfrac{28}{45}}$.

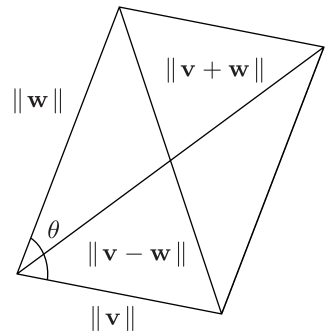
3.2.1. (a) $| \mathbf{v}_1 \cdot \mathbf{v}_2 | = 3 \leq 5 = \sqrt{5}\,\sqrt{5} = \| \mathbf{v}_1 \| \, \| \mathbf{v}_2 \|$; angle: $\cos^{-1} \frac{3}{5} \approx .9273$;

(c) $| \mathbf{v}_1 \cdot \mathbf{v}_2 | = 0 \leq 2\sqrt{6} = \sqrt{2}\,\sqrt{12} = \| \mathbf{v}_1 \| \, \| \mathbf{v}_2 \|$; angle: $\frac{1}{2}\pi \approx 1.5708$.

3.2.4. (a) $| \mathbf{v} \cdot \mathbf{w} | = 5 \leq 7.0711 = \sqrt{5}\,\sqrt{10} = \| \mathbf{v} \| \, \| \mathbf{w} \|$.

3.2.5. (b) $| \langle \mathbf{v}, \mathbf{w} \rangle | = 11 \leq 11.7473 = \sqrt{23}\,\sqrt{6} = \| \mathbf{v} \| \, \| \mathbf{w} \|$.

$\diamondsuit$ 3.2.6. Expanding $\|\mathbf{v}+\mathbf{w}\|^2 - \|\mathbf{v}-\mathbf{w}\|^2 = 4\langle \mathbf{v}, \mathbf{w}\rangle = 4\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta$.



3.2.9. Set $\mathbf{v} = (a_1, \ldots, a_n)^T$, $\mathbf{w} = (1, 1, \ldots, 1)^T$, so that Cauchy–Schwarz gives

$$|\mathbf{v}\cdot\mathbf{w}| = |a_1 + a_2 + \cdots + a_n| \le \sqrt{n}\,\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \|\mathbf{v}\|\,\|\mathbf{w}\|.$$

Equality holds if and only if $\mathbf{v} = a\,\mathbf{w}$, i.e., $a_1 = a_2 = \cdots = a_n$.

3.2.12. (a) $|\langle f, g\rangle| = 1 \le 1.03191 = \sqrt{\tfrac{1}{3}}\,\sqrt{\tfrac{1}{2}e^2 - \tfrac{1}{2}} = \|f\|\,\|g\|$;
(b) $|\langle f, g\rangle| = 2/e = .7358 \le 1.555 = \sqrt{\tfrac{2}{3}}\,\sqrt{\tfrac{1}{2}(e^2 - e^{-2})} = \|f\|\,\|g\|$.

3.2.13. (a) $\tfrac{1}{2}\pi$, (b) $\cos^{-1}\dfrac{2\sqrt{2}}{\sqrt{\pi}} = .450301$.

3.2.14. (a) $|\langle f, g\rangle| = \tfrac{2}{3} \le \sqrt{\tfrac{28}{45}} = \|f\|\,\|g\|$.

3.2.15. (a) $a = -\tfrac{4}{3}$; (b) no.

3.2.16. All scalar multiples of $\left(\tfrac{1}{2}, -\tfrac{7}{4}, 1\right)^T$.

3.2.18. All vectors in the subspace spanned by $\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\0\\1\end{pmatrix}$, so $\mathbf{v} = a\begin{pmatrix}1\\-2\\1\\0\end{pmatrix} + b\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}$.

3.2.21. (a) All solutions to $a + b = 1$.

$\diamondsuit$ 3.2.23. Choose $\mathbf{v} = \mathbf{w}$; then $0 = \langle \mathbf{w}, \mathbf{w}\rangle = \|\mathbf{w}\|^2$, and hence $\mathbf{w} = \mathbf{0}$.

3.2.26. (a) $\langle p_1, p_2\rangle = \displaystyle\int_0^1 \left(x - \tfrac{1}{2}\right)dx = 0$, $\langle p_1, p_3\rangle = \displaystyle\int_0^1 \left(x^2 - x + \tfrac{1}{6}\right)dx = 0$,
$$\langle p_2, p_3\rangle = \int_0^1 \left(x - \tfrac{1}{2}\right)\left(x^2 - x + \tfrac{1}{6}\right)dx = 0.$$

3.2.28. $p(x) = a\Big((e-1)\,x - 1\Big) + b\Big(x^2 - (e-2)\,x\Big)$ for any $a, b \in \mathbb{R}$.

3.2.32. (a) $\|\mathbf{v}_1 + \mathbf{v}_2\| = 4 \le 2\sqrt{5} = \|\mathbf{v}_1\| + \|\mathbf{v}_2\|$,
(c) $\|\mathbf{v}_1 + \mathbf{v}_2\| = \sqrt{14} \le \sqrt{2} + \sqrt{12} = \|\mathbf{v}_1\| + \|\mathbf{v}_2\|$.

3.2.33. (a) $\|\mathbf{v}_1 + \mathbf{v}_2\| = \sqrt{5} \le \sqrt{5} + \sqrt{10} = \|\mathbf{v}_1\| + \|\mathbf{v}_2\|$.

3.2.34. (b) $\| f + g \| = \sqrt{\frac{2}{3} + \frac{1}{2} e^2 + 4 e^{-1} - \frac{1}{2} e^{-2}} \approx 2.40105$

$$\leq 2.72093 \approx \sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{1}{2}(e^2 - e^{-2})} = \| f \| + \| g \|.$$

3.2.35. (a) $\| f + g \| = \sqrt{\frac{133}{45}} \approx 1.71917 \leq 2.71917 \approx 1 + \sqrt{\frac{28}{45}} = \| f \| + \| g \|.$

3.2.37. (a) $\left| \int_0^1 f(x)\, g(x)\, e^x\, dx \right| \leq \sqrt{\int_0^1 f(x)^2 e^x\, dx} \ \sqrt{\int_0^1 g(x)^2 e^x\, dx}\,,$

$$\sqrt{\int_0^1 \big[\, f(x) + g(x) \,\big]^2 e^x\, dx} \leq \sqrt{\int_0^1 f(x)^2 e^x\, dx} \ + \ \sqrt{\int_0^1 g(x)^2 e^x\, dx}\,;$$

(b) $\langle\, f , g \,\rangle = \frac{1}{2}(e^2 - 1) = 3.1945 \leq 3.3063 = \sqrt{e-1}\ \sqrt{\frac{1}{3}(e^3 - 1)} = \| f \|\, \| g \|,$

$\| f + g \| = \sqrt{\frac{1}{3} e^3 + e^2 + e - \frac{7}{3}} = 3.8038 \leq 3.8331 = \sqrt{e-1} + \sqrt{\frac{1}{3}(e^3 - 1)} = \| f \| + \| g \|;$

(c) $\cos\theta = \dfrac{\sqrt{3}}{2}\, \dfrac{e^2 - 1}{\sqrt{(e-1)(e^3-1)}} = .9662$, so $\theta = .2607$.

3.3.1. $\| \mathbf{v} + \mathbf{w} \|_1 = 2 \leq 2 = 1 + 1 = \| \mathbf{v} \|_1 + \| \mathbf{w} \|_1,$

$\| \mathbf{v} + \mathbf{w} \|_2 = \sqrt{2} \leq 2 = 1 + 1 = \| \mathbf{v} \|_2 + \| \mathbf{w} \|_2,$

$\| \mathbf{v} + \mathbf{w} \|_3 = \sqrt[3]{2} \leq 2 = 1 + 1 = \| \mathbf{v} \|_3 + \| \mathbf{w} \|_3,$

$\| \mathbf{v} + \mathbf{w} \|_\infty = 1 \leq 2 = 1 + 1 = \| \mathbf{v} \|_\infty + \| \mathbf{w} \|_\infty.$

3.3.3. (a) $\| \mathbf{u} - \mathbf{v} \|_1 = 5,\ \ \| \mathbf{u} - \mathbf{w} \|_1 = 6,\ \ \| \mathbf{v} - \mathbf{w} \|_1 = 7,$ so $\mathbf{u}, \mathbf{v}$ are closest.
(b) $\| \mathbf{u} - \mathbf{v} \|_2 = \sqrt{13},\ \ \| \mathbf{u} - \mathbf{w} \|_2 = \sqrt{12},\ \ \| \mathbf{v} - \mathbf{w} \|_2 = \sqrt{21},$ so $\mathbf{u}, \mathbf{w}$ are closest.

3.3.6. (a) $\| f - g \|_1 = \frac{1}{2} = .5,\ \ \| f - h \|_1 = 1 - \frac{2}{\pi} = .36338,\ \ \| g - h \|_1 = \frac{1}{2} - \frac{1}{\pi} = .18169,$
so g, h are closest. (b) $\| f - g \|_2 = \sqrt{\frac{1}{3}} = .57735,\ \ \| f - h \|_2 = \sqrt{\frac{3}{2} - \frac{4}{\pi}} = .47619,$
$\| g - h \|_2 = \sqrt{\frac{5}{6} - \frac{2}{\pi}} = .44352,$ so g, h are closest.

3.3.7. (a) $\| f + g \|_1 = \frac{3}{4} = .75 \leq 1.3125 \approx 1 + \frac{5}{16} = \| f \|_1 + \| g \|_1;$
(b) $\| f + g \|_2 = \sqrt{\frac{31}{48}} \approx .8036 \leq 1.3819 \approx 1 + \sqrt{\frac{7}{48}} = \| f \|_2 + \| g \|_2.$

3.3.10. (a) Comes from weighted inner product $\langle\, \mathbf{v} , \mathbf{w} \,\rangle = 2 v_1 w_1 + 3 v_2 w_2$.
(c) Clearly positive; $\| c\mathbf{v} \| = 2\,|\, c v_1 \,| + |\, c v_2 \,| = |\, c \,| \big(2\,|\, v_1 \,| + |\, v_2 \,| \big) = |\, c \,|\, \| \mathbf{v} \|;$
$\| \mathbf{v} + \mathbf{w} \| = 2\,|\, v_1 + w_1 \,| + |\, v_2 + w_2 \,| \leq 2\,|\, v_1 \,| + |\, v_2 \,| + 2\,|\, w_1 \,| + |\, w_2 \,| = \| \mathbf{v} \| + \| \mathbf{w} \|.$

3.3.11. (a) Yes; (b) no, since, for instance, $\| \,(1, -1, 0 \,)^T \,\| = 0$.

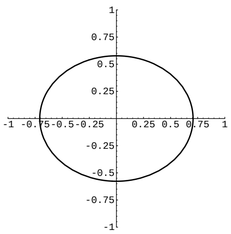
3.3.13. True for an inner product norm, but false in general. For example,
$$\| \mathbf{e}_1 + \mathbf{e}_2 \|_1 = 2 = \| \mathbf{e}_1 \|_1 + \| \mathbf{e}_2 \|_1.$$

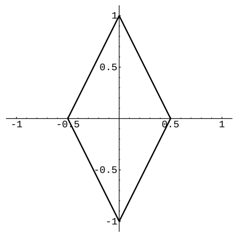
♢ 3.3.17. (a) $\| f + g \|_1 = \int_a^b |\, f(x) + g(x) \,|\, dx \leq \int_a^b \big[\, |\, f(x) \,| + |\, g(x) \,| \,\big]\, dx$

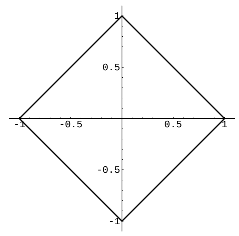
$$= \int_a^b |\, f(x) \,|\, dx + \int_a^b |\, g(x) \,|\, dx = \| f \|_1 + \| g \|_1.$$

3.3.20. (b) $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -1 \end{pmatrix}$; (d) $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -1 \end{pmatrix}$.

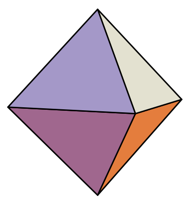
3.3.22. (b) 2 vectors, namely $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$ and $-\mathbf{u} = -\mathbf{v}/\|\mathbf{v}\|$.

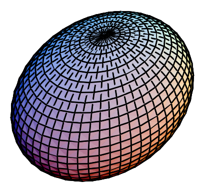
3.3.24. (a) 

(c) 

(e) 

3.3.25.

(a) Unit octahedron: 

(c) Ellipsoid with semi-axes $\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{3}}$: 

3.3.28. (a) $\frac{18}{5}\,x - \frac{6}{5}$; (c) $\frac{3}{2}\,x - \frac{1}{2}$; (e) $\frac{3}{2\sqrt{2}}\,x - \frac{1}{2\sqrt{2}}$.

3.3.29. (a) Yes, (c) yes, (e) no, (g) no — its norm is not defined.

3.3.31. (a) $\|\mathbf{v}\|_2 = \sqrt{2}$, $\|\mathbf{v}\|_\infty = 1$, and $\frac{1}{\sqrt{2}}\sqrt{2} \le 1 \le \sqrt{2}$;

(c) $\|\mathbf{v}\|_2 = 2$, $\|\mathbf{v}\|_\infty = 1$, and $\frac{1}{2}\,2 \le 1 \le 2$.

3.3.32. (a) $\mathbf{v} = (a, 0)^T$ or $(0, a)^T$; (c) $\mathbf{v} = (a, 0)^T$ or $(0, a)^T$.

3.3.35. (i)
$$\|\mathbf{v}\|_1^2 = \left(\sum_{i=1}^n |v_i|\right)^2 = \sum_{i=1}^n |v_i|^2 + 2\sum_{i<j} |v_i|\,|v_j| \ge \sum_{i=1}^n |v_i|^2 = \|\mathbf{v}\|_2^2.$$

On the other hand, since $2xy \le x^2 + y^2$,
$$\|\mathbf{v}\|_1^2 = \sum_{i=1}^n |v_i|^2 + 2\sum_{i<j} |v_i|\,|v_j| \le n \sum_{i=1}^n |v_i|^2 = n\,\|\mathbf{v}\|_2^2.$$

(ii) (a) $\|\mathbf{v}\|_2 = \sqrt{2}$, $\|\mathbf{v}\|_1 = 2$, and $\sqrt{2} \le 2 \le \sqrt{2}\,\sqrt{2}$.

(iii) (a) $\mathbf{v} = c\,\mathbf{e}_j$ for some $j = 1, \ldots, n$.

3.3.37. In each case, we minimize and maximize $\|\,(\cos\theta, \sin\theta\,)^T\,\|$ for $0 \le \theta \le 2\pi$:
(a) $c^\star = \sqrt{2}\,,\ \ C^\star = \sqrt{3}\,.$

♡ 3.3.40. (a) The maximum (absolute) value of $f_n(x)$ is $1 = \|\,f_n\,\|_\infty$. On the other hand,

$$\|\,f_n\,\|_2 = \sqrt{\int_{-\infty}^{\infty} |\,f_n(x)\,|^2\,dx} \;=\; \sqrt{\int_{-n}^{n} dx} = \sqrt{2n} \qquad \longrightarrow \qquad \infty.$$

(b) Suppose there exists a constant C such that $\|\,f\,\|_2 \le C\,\|\,f\,\|_\infty$ for all functions. Then, in particular, $\sqrt{2n} = \|\,f_n\,\|_2 \le C\,\|\,f_n\,\|_\infty = C$ for all n, which is impossible.

3.3.45. (a) $\frac{3}{4}$, (c) .9.

3.3.47. False: For instance, if $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, then $B = S^{-1} A\, S = \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix}$, and $\|\,B\,\|_\infty = 2 \ne 1 = \|\,A\,\|_\infty$.

♢ 3.3.48. (i) The 1 matrix norm is the maximum absolute column sum:

$$\|\,A\,\|_1 = \max\left\{ \sum_{i=1}^{n} |\,a_{ij}\,| \;\middle|\; 1 \le j \le n \right\}.$$

(ii) (a) $\frac{5}{6}$, (c) $\frac{8}{7}$.

3.4.1. (a) Positive definite: $\langle\, \mathbf{v}\,, \mathbf{w}\,\rangle = v_1 w_1 + 2 v_2 w_2$; (c) not positive definite;
(e) positive definite: $\langle\, \mathbf{v}\,, \mathbf{w}\,\rangle = v_1 w_1 - v_1 w_2 - v_2 w_1 + 3 v_2 w_2$.

♢ 3.4.5. (a) $k_{ii} = \mathbf{e}_i^T K\, \mathbf{e}_i > 0$. (b) For example, $K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is not positive definite or even semi-definite. (c) For example, $K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

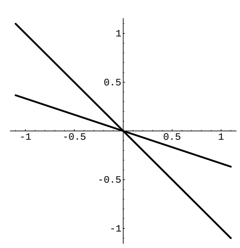
3.4.8. For example, $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 2 & 5 \end{pmatrix}$ is not even symmetric. Even the associated quadratic form $(\, x \quad y\,) \begin{pmatrix} 4 & 7 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4x^2 + 9xy + 5y^2$ is not positive definite.

♢ 3.4.12. If $q(\mathbf{x}) = \mathbf{x}^T K\, \mathbf{x}$ with $K^T = K$, then

$$q(\mathbf{x}+\mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y}) = (\mathbf{x}+\mathbf{y})^T K (\mathbf{x}+\mathbf{y}) - \mathbf{x}^T K\, \mathbf{x} - \mathbf{y}^T K\, \mathbf{y} = 2\mathbf{x}^T K\, \mathbf{y} = 2\,\langle\, \mathbf{x}\,, \mathbf{y}\,\rangle.$$

3.4.17. $\mathbf{x}^T K\, \mathbf{x} = (\,1 \quad 1\,) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \mathbf{0}$, but $K\,\mathbf{x} = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \ne \mathbf{0}$.

♢ 3.4.20. (b) $x^2 + 4xy + 3y^2 = (x+y)(x+3y) = 0$:



3.4.22. (*i*) $\begin{pmatrix} 10 & 6 \\ 6 & 4 \end{pmatrix}$; positive definite. (*iii*) $\begin{pmatrix} 6 & -8 \\ -8 & 13 \end{pmatrix}$; positive definite. (*v*) $\begin{pmatrix} 9 & 6 & 3 \\ 6 & 6 & 0 \\ 3 & 0 & 3 \end{pmatrix}$;

positive semi-definite; null directions: all nonzero scalar multiples of $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

3.4.23. (*iii*) $\begin{pmatrix} 9 & -12 \\ -12 & 21 \end{pmatrix}$, positive definite; (*v*) $\begin{pmatrix} 21 & 12 & 9 \\ 12 & 9 & 3 \\ 9 & 3 & 6 \end{pmatrix}$, positive semi-definite.

Note: Positive definiteness doesn't change, since it only depends upon the linear independence of the vectors.

3.4.25. $K = \begin{pmatrix} 1 & e-1 & \frac{1}{2}(e^2-1) \\ e-1 & \frac{1}{2}(e^2-1) & \frac{1}{3}(e^3-1) \\ \frac{1}{2}(e^2-1) & \frac{1}{3}(e^3-1) & \frac{1}{4}(e^4-1) \end{pmatrix}$

is positive definite since $1, e^x, e^{2x}$ are linearly independent functions.

♢ 3.4.30. (*a*) is a special case of (*b*) since positive definite matrices are symmetric.

(*b*) By Theorem 3.34 if S is *any* symmetric matrix, then $S^T S = S^2$ is always positive semi-definite, and positive definite if and only if $\ker S = \{\mathbf{0}\}$, i.e., S is nonsingular. In particular, if $S = K > 0$, then $\ker K = \{\mathbf{0}\}$ and so $K^2 > 0$.

♢ 3.4.33. Let $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y}$ be the corresponding inner product. Then $k_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$, and hence K is the Gram matrix associated with the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$.

3.5.1. (a) Positive definite; (*c*) not positive definite; (*e*) positive definite.

3.5.2. (a) $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; not positive definite.

(*c*) $\begin{pmatrix} 3 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 1 & \frac{3}{7} & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{14}{3} & 0 \\ 0 & 0 & \frac{8}{7} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{3} & 1 \\ 0 & 1 & \frac{3}{7} \\ 0 & 0 & 1 \end{pmatrix}$; positive definite.

3.5.4. $K = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 2 & 0 \\ -\frac{1}{2} & 0 & 3 \end{pmatrix}$; yes, it is positive definite.

3.5.5. (a) $(x + 4y)^2 - 15y^2$; not positive definite. (*b*) $(x - 2y)^2 + 3y^2$; positive definite.

3.5.6. (a) $(x + 2z)^2 + 3y^2 + z^2$.

3.5.7. (a) $\begin{pmatrix} x & y & z \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; not positive definite;

(c) $\begin{pmatrix} x & y & z \end{pmatrix}^T \begin{pmatrix} 1 & 1 & -2 \\ 1 & 2 & -3 \\ -2 & -3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; positive definite.

3.5.10. (b) $\operatorname{tr} K = \sum_{i=1}^{n} k_{ii} > 0$ since, according to Exercise 3.4.5, every diagonal entry of K is positive.

3.5.13. Write $S = (S + c\,\mathrm{I}) + (-c\,\mathrm{I}) = K + N$, where $N = -c\,\mathrm{I}$ is negative definite for any $c > 0$, while $K = S + c\,\mathrm{I}$ is positive definite provided $c \gg 0$ is sufficiently large.

3.5.19. (b) $\begin{pmatrix} 4 & -12 \\ -12 & 45 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -6 & 3 \end{pmatrix} \begin{pmatrix} 2 & -6 \\ 0 & 3 \end{pmatrix}$,

(d) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}$.

3.5.20. (a) $\begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & \sqrt{3} \end{pmatrix}$, (c) no factorization.

3.5.21. (b) $z_1^2 + z_2^2$, where $z_1 = x_1 - x_2$, $z_2 = \sqrt{3}\,x_2$;

(d) $z_1^2 + z_2^2 + z_3^2$, where $z_1 = \sqrt{3}\,x_1 - \frac{1}{\sqrt{3}}\,x_2 - \frac{1}{\sqrt{3}}\,x_3$, $z_2 = \sqrt{\frac{5}{3}}\,x_2 - \frac{1}{\sqrt{15}}\,x_3$, $z_3 = \sqrt{\frac{28}{5}}\,x_3$.

3.6.2. $e^{k\pi\,\mathrm{i}} = \cos k\pi + \mathrm{i}\,\sin k\pi = (-1)^k = \begin{cases} 1, & k \text{ even}, \\ -1, & k \text{ odd}. \end{cases}$

3.6.5. (a) $\mathrm{i} = e^{\pi\,\mathrm{i}/2}$; (b) $\sqrt{\mathrm{i}} = e^{\pi\,\mathrm{i}/4} = \dfrac{1}{\sqrt{2}} + \dfrac{\mathrm{i}}{\sqrt{2}}$ and $e^{5\pi\,\mathrm{i}/4} = -\dfrac{1}{\sqrt{2}} - \dfrac{\mathrm{i}}{\sqrt{2}}$.

3.6.7. (a) $1/z$ moves in a clockwise direction around a circle of radius $1/r$.

♢ 3.6.9. Write $z = r\,e^{\mathrm{i}\,\theta}$ so $\theta = \operatorname{ph} z$. Then $\operatorname{Re} e^{\mathrm{i}\,\varphi} z = \operatorname{Re}\,(r\,e^{\mathrm{i}\,(\varphi+\theta)}) = r\cos(\varphi+\theta) \le r = |\,z\,|$, with equality if and only if $\varphi + \theta$ is an integer multiple of 2π.

3.6.13. Set $z = x + \mathrm{i}\,y$, $w = u + \mathrm{i}\,v$, then $z\,\overline{w} = (x + \mathrm{i}\,y)\cdot(u - \mathrm{i}\,v) = (x\,u + y\,v) + \mathrm{i}\,(y\,u - x\,v)$ has real part $\operatorname{Re}\,(z\,\overline{w}) = x\,u + y\,v$, which is the dot product between $(\,x, y\,)^T$ and $(\,u, v\,)^T$.

3.6.16. (b) $\cos 3\theta = \cos^3\theta - 3\cos\theta\,\sin^2\theta$, $\sin 3\theta = 3\cos\theta\,\sin^2\theta - \sin^3\theta$.

3.6.18. $e^z = e^x\cos y + \mathrm{i}\,e^x \sin y = r\cos\theta + \mathrm{i}\,r\sin\theta$ implies $r = |\,e^z\,| = e^x$ and $\theta = \operatorname{ph} e^z = y$.

♢ 3.6.22. $x^{a+\mathrm{i}\,b} = x^a\,e^{\mathrm{i}\,b\log x} = x^a\cos(b\log x) + \mathrm{i}\,x^a\sin(b\log x)$.

♢ 3.6.24. (a) $\dfrac{d}{dx}\, e^{\lambda x} = \dfrac{d}{dx}\left(e^{\mu x}\cos \nu x + \mathrm{i}\, e^{\mu x}\sin \nu x \right) = (\mu\, e^{\mu x}\cos \nu x - \nu\, e^{\mu x}\sin \nu x) +$
$+ \,\mathrm{i}\,(\mu e^{\mu x}\sin \nu x + \nu\, e^{\mu x}\cos \nu x) = (\mu + \mathrm{i}\,\nu)\left(e^{\mu x}\cos \nu x + \mathrm{i}\, e^{\mu x}\sin \nu x \right) = \lambda\, e^{\lambda x}.$

3.6.25. (a) $\frac12 x + \frac14 \sin 2x$, (c) $-\frac14 \cos 2x$, (e) $\frac38 x + \frac14 \sin 2x + \frac{1}{32}\sin 4x$.

3.6.26. (b) Linearly dependent; (d) linearly dependent; (f) linearly independent.

3.6.27. False — it is not closed under scalar multiplication. For instance, $\mathrm{i}\begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \begin{pmatrix} \mathrm{i}\, z \\ \mathrm{i}\, \overline{z} \end{pmatrix}$ is not in the subspace since $\overline{\mathrm{i}\, z} = -\,\mathrm{i}\, \overline{z}$.

3.6.29. (b) Dimension $= 2$; basis: $(\,\mathrm{i} - 1, 0, 1\,)^T,\ (\,-\,\mathrm{i}, 1, 0\,)^T$.
(d) Dimension $= 1$; basis: $\left(-\frac{14}{5} - \frac85\,\mathrm{i}, \frac{13}{5} - \frac45\,\mathrm{i}, 1 \right)^T$.

3.6.30. (b) Image: $\begin{pmatrix} 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -1+\mathrm{i} \\ 3 \end{pmatrix}$; coimage: $\begin{pmatrix} 2 \\ -1+\mathrm{i} \\ 1-2\,\mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 \\ 1+\mathrm{i} \\ 3-3\,\mathrm{i} \end{pmatrix}$; kernel: $\begin{pmatrix} 1+\frac52\,\mathrm{i} \\ 3\,\mathrm{i} \\ 1 \end{pmatrix}$;
cokernel: $\{\mathbf{0}\}$.

3.6.33. (a) Not a subspace; (b) subspace; (d) not a subspace.

3.6.35. (a) Belongs: $\sin x = -\frac12\,\mathrm{i}\, e^{\mathrm{i}\,x} + \frac12\,\mathrm{i}\, e^{-\,\mathrm{i}\,x}$; (c) doesn't belong;
(d) belongs: $\sin^2 \frac12 x = \frac12 - \frac12 e^{\mathrm{i}\,x} - \frac12 e^{-\,\mathrm{i}\,x}$.

3.6.36. (a) Sesquilinearity:
$$\begin{aligned} \langle\, c\,\mathbf{u} + d\,\mathbf{v}\,, \mathbf{w}\,\rangle &= (c u_1 + d v_1)\overline{w}_1 + 2\,(c u_2 + d v_2)\overline{w}_2 \\ &= c\,(u_1\,\overline{w}_1 + 2\,u_2\,\overline{w}_2) + d\,(v_1\,\overline{w}_1 + 2\,v_2\,\overline{w}_2) = c\,\langle\, \mathbf{u}\,, \mathbf{w}\,\rangle + d\,\langle\, \mathbf{v}\,, \mathbf{w}\,\rangle, \\ \langle\, \mathbf{u}\,, c\,\mathbf{v} + d\,\mathbf{w}\,\rangle &= u_1\,(\,\overline{c v_1 + d w_1}\,) + 2\,u_2\,(\,\overline{c v_2 + d w_2}\,) \\ &= \bar{c}\,(u_1\,\overline{v}_1 + 2\,u_2\,\overline{v}_2) + \bar{d}\,(u_1\,\overline{w}_1 + 2\,u_2\,\overline{w}_2) = \bar{c}\,\langle\, \mathbf{u}\,, \mathbf{v}\,\rangle + \bar{d}\,\langle\, \mathbf{u}\,, \mathbf{w}\,\rangle. \end{aligned}$$
Conjugate Symmetry:
$$\langle\, \mathbf{v}\,, \mathbf{w}\,\rangle = v_1\,\overline{w}_1 + 2\,v_2\,\overline{w}_2 = \overline{w_1\,\overline{v}_1 + 2\,w_2\,\overline{v}_2} = \overline{\langle\, \mathbf{w}\,, \mathbf{v}\,\rangle}.$$
Positive definite: $\langle\, \mathbf{v}\,, \mathbf{v}\,\rangle = |\,v_1\,|^2 + 2\,|\,v_2\,|^2 > 0$ whenever $\mathbf{v} = (\,v_1, v_2\,)^T \neq \mathbf{0}$.

3.6.37. (a) No, (b) no, (d) yes.

♢ 3.6.40. (a)
$$\begin{aligned} \|\,\mathbf{z} + \mathbf{w}\,\|^2 &= \langle\, \mathbf{z} + \mathbf{w}\,, \mathbf{z} + \mathbf{w}\,\rangle = \|\,\mathbf{z}\,\|^2 + \langle\, \mathbf{z}\,, \mathbf{w}\,\rangle + \langle\, \mathbf{w}\,, \mathbf{z}\,\rangle + \|\,\mathbf{w}\,\|^2 \\ &= \|\,\mathbf{z}\,\|^2 + \langle\, \mathbf{z}\,, \mathbf{w}\,\rangle + \overline{\langle\, \mathbf{z}\,, \mathbf{w}\,\rangle} + \|\,\mathbf{w}\,\|^2 = \|\,\mathbf{z}\,\|^2 + 2\,\mathrm{Re}\,\langle\, \mathbf{z}\,, \mathbf{w}\,\rangle + \|\,\mathbf{w}\,\|^2. \end{aligned}$$

3.6.46. (e) Infinitely many, namely $\mathbf{u} = e^{\mathrm{i}\,\theta}\mathbf{v}/\|\,\mathbf{v}\,\|$ for any $0 \le \theta < 2\pi$.

3.6.48. (b) (i) $\langle\, x + \mathrm{i}\,, x - \mathrm{i}\,\rangle = -\frac23 + \mathrm{i}$, $\|\,x + \mathrm{i}\,\| = \|\,x - \mathrm{i}\,\| = \frac{2}{\sqrt3}$;
(ii) $|\,\langle\, x + \mathrm{i}\,, x - \mathrm{i}\,\rangle\,| = \frac{\sqrt{13}}{3} \le \frac43 = \|\,x + \mathrm{i}\,\|\,\|\,x - \mathrm{i}\,\|$,
$\|\,(x + \mathrm{i}\,) + (x - \mathrm{i}\,)\,\| = \|\,2\,x\,\| = \frac{2}{\sqrt3} \le \frac{4}{\sqrt3} = \|\,x + \mathrm{i}\,\| + \|\,x - \mathrm{i}\,\|$.

Students' Solutions Manual for

Chapter 4: Orthogonality

4.1.1. (a) Orthogonal basis; (c) not a basis; (e) orthogonal basis.

4.1.2. (b) Orthonormal basis.

4.1.3. (a) Basis; (c) not a basis; (e) orthonormal basis.

4.1.5. (a) $a = \pm 1$.

4.1.6. $a = 2b > 0$.

4.1.9. False. Consider the basis $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Under the weighted inner product, $\langle \mathbf{v}_1 , \mathbf{v}_2 \rangle = b > 0$, since the coefficients of a, b, c appearing in the inner product must be strictly positive.

♡ 4.1.13. (a) The (i, j) entry of $A^T K A$ is $\mathbf{v}_i^T K \mathbf{v}_j = \langle \mathbf{v}_i , \mathbf{v}_j \rangle$. Thus, $A^T K A = \mathrm{I}$ if and only if

$$\langle \mathbf{v}_i , \mathbf{v}_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad \text{and so the vectors form an orthonormal basis.}$$

(b) According to part (a), orthonormality requires $A^T K A = \mathrm{I}$, and so $K = A^{-T} A^{-1} = (A A^T)^{-1}$ is the Gram matrix for A^{-1}, and $K > 0$ since A^{-1} is nonsingular. This also proves the uniqueness of the inner product.

(c) $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $K = \begin{pmatrix} 10 & -7 \\ -7 & 5 \end{pmatrix}$, with inner product

$$\langle \mathbf{v} , \mathbf{w} \rangle = \mathbf{v}^T K \mathbf{w} = 10 v_1 w_1 - 7 v_1 w_2 - 7 v_2 w_1 + 5 v_2 w_2 .$$

4.1.16. $\langle \mathbf{v}_1 + \mathbf{v}_2 , \mathbf{v}_1 - \mathbf{v}_2 \rangle = \| \mathbf{v}_1 \|^2 - \| \mathbf{v}_2 \|^2 = 0$ by assumption. Moreover, since $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, neither $\mathbf{v}_1 - \mathbf{v}_2$ nor $\mathbf{v}_1 + \mathbf{v}_2$ is zero, and hence Theorem 4.5 implies that they form an orthogonal basis for the two-dimensional vector space V.

4.1.18. (b) $p(t) = c \left(1 - \frac{3}{2} t \right)$ for any c.

4.1.22. (a) We compute $\langle \mathbf{v}_1 , \mathbf{v}_2 \rangle = \langle \mathbf{v}_1 , \mathbf{v}_3 \rangle = \langle \mathbf{v}_2 , \mathbf{v}_3 \rangle = 0$ and $\| \mathbf{v}_1 \| = \| \mathbf{v}_2 \| = \| \mathbf{v}_3 \| = 1$.

(b) $\langle \mathbf{v} , \mathbf{v}_1 \rangle = \frac{7}{5}$, $\langle \mathbf{v} , \mathbf{v}_2 \rangle = \frac{11}{13}$, and $\langle \mathbf{v} , \mathbf{v}_3 \rangle = -\frac{37}{65}$, so $(1, 1, 1)^T = \frac{7}{5} \mathbf{v}_1 + \frac{11}{13} \mathbf{v}_2 - \frac{37}{65} \mathbf{v}_3$.

(c) $\left(\frac{7}{5} \right)^2 + \left(\frac{11}{13} \right)^2 + \left(-\frac{37}{65} \right)^2 = 3 = \| \mathbf{v} \|^2$.

4.1.23. (a) Because $\langle \mathbf{v}_1 , \mathbf{v}_2 \rangle = \mathbf{v}_1^T K \mathbf{v}_2 = 0$.

(b) $\mathbf{v} = \dfrac{\langle \mathbf{v} , \mathbf{v}_1 \rangle}{\| \mathbf{v}_1 \|^2} \mathbf{v}_1 + \dfrac{\langle \mathbf{v} , \mathbf{v}_2 \rangle}{\| \mathbf{v}_2 \|^2} \mathbf{v}_2 = \frac{7}{3} \mathbf{v}_1 - \frac{1}{3} \mathbf{v}_2$.

(c) $\left(\dfrac{\mathbf{v}_1 \cdot \mathbf{v}}{\| \mathbf{v}_1 \|} \right)^2 + \left(\dfrac{\mathbf{v}_2 \cdot \mathbf{v}}{\| \mathbf{v}_2 \|} \right)^2 = \left(\frac{7}{\sqrt{3}} \right)^2 + \left(-\frac{5}{\sqrt{15}} \right)^2 = 18 = \| \mathbf{v} \|^2$.

(*d*) $\mathbf{u}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)^T$, $\mathbf{u}_2 = \left(-\frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}} \right)^T$.

(*e*) $\mathbf{v} = \langle \mathbf{v}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{v}, \mathbf{u}_2 \rangle \mathbf{u}_2 = \frac{7}{\sqrt{3}} \mathbf{u}_1 - \frac{\sqrt{15}}{3} \mathbf{u}_2$; $\| \mathbf{v} \|^2 = 18 = \left(\frac{7}{\sqrt{3}} \right)^2 + \left(-\frac{\sqrt{15}}{3} \right)^2$.

4.1.26. $\dfrac{\langle x, p_1 \rangle}{\| p_1 \|^2} = \dfrac{1}{2}$, $\dfrac{\langle x, p_2 \rangle}{\| p_2 \|^2} = 1$, $\dfrac{\langle x, p_3 \rangle}{\| p_3 \|^2} = 0$, so $x = \frac{1}{2} p_1(x) + p_2(x)$.

4.1.29. (*b*) $\cos x \sin x = \frac{1}{2} \sin 2x$, (*d*) $\cos^2 x \sin^3 x = \frac{1}{8} \sin x + \frac{1}{16} \sin 3x - \frac{1}{16} \sin 5x$.

♢ 4.1.31. $\langle e^{\,i\,k\,x}, e^{\,i\,l\,x} \rangle = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} e^{\,i\,k\,x}\, \overline{e^{\,i\,k\,x}}\, dx = \dfrac{1}{2\pi} \int_{-\pi}^{\pi} e^{\,i\,(k-l)\,x}\, dx = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$

4.2.1. (*b*) $\frac{1}{\sqrt{2}} (1, 1, 0)^T$, $\frac{1}{\sqrt{6}} (-1, 1, -2)^T$, $\frac{1}{\sqrt{3}} (1, -1, -1)^T$.

4.2.2. (*a*) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0 \right)^T$, $\left(0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right)^T$, $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)^T$, $\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)^T$.

4.2.4. (*b*) Starting with the basis $\left(\frac{1}{2}, 1, 0 \right)^T$, $(-1, 0, 1)^T$, the Gram–Schmidt process produces the orthonormal basis $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \right)^T$, $\left(-\frac{4}{3\sqrt{5}}, \frac{2}{3\sqrt{5}}, \frac{5}{3\sqrt{5}} \right)^T$.

4.2.6. (*a*) $\frac{1}{\sqrt{3}} (1, 1, -1, 0)^T$, $\frac{1}{\sqrt{15}} (-1, 2, 1, 3)^T$, $\frac{1}{\sqrt{15}} (3, -1, 2, 1)^T$.

(*c*) Applying Gram-Schmidt to the coimage basis $(2, 1, 0, -1)^T$, $\left(0, \frac{1}{2}, -1, \frac{1}{2} \right)^T$, gives the orthonormal basis $\frac{1}{\sqrt{6}} (2, 1, 0, -1)^T$, $\frac{1}{\sqrt{6}} (0, 1, -2, 1)^T$.

(*e*) Applying Gram-Schmidt to the cokernel basis $\left(\frac{2}{3}, -\frac{1}{3}, 1, 0 \right)^T$, $(-4, 3, 0, 1)^T$, gives the orthonormal basis $\frac{1}{\sqrt{14}} (2, -1, 3, 0)^T$, $\frac{1}{9\sqrt{42}} (-34, 31, 33, 14)^T$.

4.2.7. (*a*) Image: $\dfrac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$; kernel: $\dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; coimage: $\dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$; cokernel: $\dfrac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

(*c*) Image: $\dfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\dfrac{1}{\sqrt{42}} \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$, $\dfrac{1}{\sqrt{14}} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$; kernel: $\dfrac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$;

coimage: $\dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\dfrac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\dfrac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$; the cokernel is $\{\mathbf{0}\}$, so there is no basis.

4.2.8. Applying the Gram–Schmidt process to the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$ gives

(*a*) $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \frac{1}{\sqrt{5}} \end{pmatrix}$; (*b*) $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{2\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{pmatrix}$.

4.2.10. (*i*) (*b*) $\frac{1}{\sqrt{5}} (1, 1, 0)^T$, $\frac{1}{\sqrt{55}} (-2, 3, -5)^T$, $\frac{1}{\sqrt{66}} (2, -3, -6)^T$;

(*ii*) (*b*) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)^T$, $\left(-\frac{1}{2}, 0, -\frac{1}{2} \right)^T$, $\left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)^T$.

4.2.15. (a) $\left(\dfrac{1+\mathrm{i}}{2}, \dfrac{1-\mathrm{i}}{2} \right)^T, \quad \left(\dfrac{3-\mathrm{i}}{2\sqrt{5}}, \dfrac{1+3\,\mathrm{i}}{2\sqrt{5}} \right)^T.$

4.2.16. (a) $\left(\dfrac{1-\mathrm{i}}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, 0 \right)^T, \quad \left(\dfrac{-1+2\,\mathrm{i}}{2\sqrt{6}}, \dfrac{3-\mathrm{i}}{2\sqrt{6}}, \dfrac{3\,\mathrm{i}}{2\sqrt{6}} \right)^T.$

4.2.17. (a) $\begin{pmatrix} 0. \\ .7071 \\ .7071 \end{pmatrix}, \begin{pmatrix} .8165 \\ -.4082 \\ .4082 \end{pmatrix}, \begin{pmatrix} .57735 \\ .57735 \\ -.57735 \end{pmatrix};$ (c) $\begin{pmatrix} .5164 \\ .2582 \\ .7746 \\ .2582 \\ 0. \end{pmatrix}, \begin{pmatrix} -.2189 \\ -.5200 \\ .4926 \\ -.5200 \\ .4105 \end{pmatrix}, \begin{pmatrix} .2529 \\ .5454 \\ -.2380 \\ -.3372 \\ .6843 \end{pmatrix}.$

4.2.21. Since $\mathbf{u}_1, \ldots, \mathbf{u}_n$ form an orthonormal basis, if $i < j$,

$$\langle \mathbf{w}_k^{(j+1)}, \mathbf{u}_i \rangle = \langle \mathbf{w}_k^{(j)}, \mathbf{u}_i \rangle,$$

and hence, by induction, $r_{ik} = \langle \mathbf{w}_k, \mathbf{u}_i \rangle = \langle \mathbf{w}_k^{(i)}, \mathbf{u}_i \rangle$. Furthermore,

$$\| \mathbf{w}_k^{(j+1)} \|^2 = \| \mathbf{w}_k^{(j)} \|^2 - \langle \mathbf{w}_k^{(j)}, \mathbf{u}_j \rangle^2 = \| \mathbf{w}_k^{(j)} \|^2 - r_{jk}^2,$$

and so, by (4.5), $\| \mathbf{w}_i^{(i)} \|^2 = \| \mathbf{w}_i \|^2 - r_{1i}^2 - \cdots - r_{i-1,i}^2 = r_{ii}^2$.

4.3.1. (a) Neither; (c) orthogonal; (f) proper orthogonal.

4.3.3. (a) True: Using the formula (4.31) for an improper 2×2 orthogonal matrix,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

4.3.7. $(Q^{-1})^T = (Q^T)^T = Q = (Q^{-1})^{-1}$, proving orthogonality.

4.3.9. In general, $\det(Q_1 Q_2) = \det Q_1 \det Q_2$. If both determinants are $+1$, so is their product. Improper times proper is improper, while improper times improper is proper.

4.3.14. False. This is true only for row interchanges or multiplication of a row by -1.

♢ 4.3.16. (a) $\| Q\mathbf{x} \|^2 = (Q\mathbf{x})^T Q\mathbf{x} = \mathbf{x}^T Q^T Q \mathbf{x} = \mathbf{x}^T \mathrm{I}\, \mathbf{x} = \mathbf{x}^T \mathbf{x} = \| \mathbf{x} \|^2$.

4.3.18. (a) If $S = (\, \mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_n \,)$, then $S^{-1} = S^T D$, where $D = \operatorname{diag}(1/\|\mathbf{v}_1\|^2, \ldots, 1/\|\mathbf{v}_n\|^2)$.

(b) $$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac14 & \frac14 & \frac14 & \frac14 \\ \frac14 & \frac14 & -\frac14 & -\frac14 \\ \frac12 & -\frac12 & 0 & 0 \\ 0 & 0 & \frac12 & -\frac12 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \frac14 & 0 & 0 & 0 \\ 0 & \frac14 & 0 & 0 \\ 0 & 0 & \frac12 & 0 \\ 0 & 0 & 0 & \frac12 \end{pmatrix}.$$

♢ 4.3.23. If $S = (\, \mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_n \,)$, then the (i,j) entry of $S^T K S$ is $\mathbf{v}_i^T K \mathbf{v}_j = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$, so $S^T K S = \mathrm{I}$ if and only if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$, while $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = \| \mathbf{v}_i \|^2 = 1$.

4.3.27. (a) $\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt5} & -\frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{pmatrix} \begin{pmatrix} \sqrt5 & -\frac{1}{\sqrt5} \\ 0 & \frac{7}{\sqrt5} \end{pmatrix}$,

(c) $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt6} \\ 0 & \sqrt{\frac56} & \frac{1}{\sqrt6} \\ -\frac{1}{\sqrt5} & -\sqrt{\frac{2}{15}} & \sqrt{\frac23} \end{pmatrix} \begin{pmatrix} \sqrt5 & \frac{3}{\sqrt5} & -\frac{3}{\sqrt5} \\ 0 & \sqrt{\frac65} & 7\sqrt{\frac{2}{15}} \\ 0 & 0 & 2\sqrt{\frac23} \end{pmatrix}$,

(e) $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 4 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

4.3.28.

(ii) (a) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac23 & \frac{1}{\sqrt2} & -\frac{1}{3\sqrt2} \\ \frac13 & 0 & \frac{2\sqrt2}{3} \\ \frac23 & -\frac{1}{\sqrt2} & -\frac{1}{3\sqrt2} \end{pmatrix} \begin{pmatrix} 3 & 0 & 2 \\ 0 & \sqrt2 & -2\sqrt2 \\ 0 & 0 & \sqrt2 \end{pmatrix}$, (b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$;

♠ 4.3.29. 3×3 case:

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} .9701 & -.2339 & .0643 \\ .2425 & .9354 & -.2571 \\ 0 & .2650 & .9642 \end{pmatrix} \begin{pmatrix} 4.1231 & 1.9403 & .2425 \\ 0 & 3.773 & 1.9956 \\ 0 & 0 & 3.5998 \end{pmatrix}.$$

♡ 4.3.32. (a) If rank $A = n$, then the columns $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of A are linearly independent, and so form a basis for its image. Applying the Gram–Schmidt process converts the column basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$ to an orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of img A.

(c) (i) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt5} & -\frac23 \\ \frac{2}{\sqrt5} & \frac13 \\ 0 & \frac23 \end{pmatrix} \begin{pmatrix} \sqrt5 & \sqrt5 \\ 0 & 3 \end{pmatrix}$.

4.3.34. (a) (i) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, (iii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. (b) (i) $\mathbf{v} = c\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, (iii) $\mathbf{v} = c\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

4.3.35. Exercise 4.3.27: (a) $\widehat{\mathbf{v}}_1 = \begin{pmatrix} -1.2361 \\ 2.0000 \end{pmatrix}$, $H_1 = \begin{pmatrix} .4472 & .8944 \\ .8944 & -.4472 \end{pmatrix}$,

$Q = \begin{pmatrix} .4472 & .8944 \\ .8944 & -.4472 \end{pmatrix}$, $R = \begin{pmatrix} 2.2361 & -.4472 \\ 0 & -3.1305 \end{pmatrix}$;

(c) $\widehat{\mathbf{v}}_1 = \begin{pmatrix} -.2361 \\ 0 \\ -1 \end{pmatrix}$, $H_1 = \begin{pmatrix} .8944 & 0 & -.4472 \\ 0 & 1 & 0 \\ -.4472 & 0 & -.8944 \end{pmatrix}$,

$\widehat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ -.0954 \\ .4472 \end{pmatrix}$, $H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & .9129 & .4082 \\ 0 & .4082 & -.9129 \end{pmatrix}$,

$Q = \begin{pmatrix} .8944 & -.1826 & .4082 \\ 0 & .9129 & .4082 \\ -.4472 & -.3651 & .8165 \end{pmatrix}$, $R = \begin{pmatrix} 2.2361 & 1.3416 & -1.3416 \\ 0 & 1.0954 & 2.556 \\ 0 & 0 & 1.633 \end{pmatrix}$;

(e) $\quad \widehat{\mathbf{v}}_1 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \widehat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$

$$Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$

Exercise 4.3.29: 3×3 case:

$$\widehat{\mathbf{v}}_1 = \begin{pmatrix} -.1231 \\ 1 \\ 0 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} .9701 & .2425 & 0 \\ .2425 & -.9701 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ -7.411 \\ 1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -.9642 & .2650 \\ 0 & .2650 & .9642 \end{pmatrix}$$

$$Q = \begin{pmatrix} .9701 & -.2339 & .0643 \\ .2425 & .9354 & -.2571 \\ 0 & .2650 & .9642 \end{pmatrix}, \qquad R = \begin{pmatrix} 4.1231 & 1.9403 & .2425 \\ 0 & 3.773 & 1.9956 \\ 0 & 0 & 3.5998 \end{pmatrix};$$

4.4.1. (a) $\mathbf{v}_2, \mathbf{v}_4$, $\quad (c)$ $\mathbf{v}_2$, $\quad (e)$ $\mathbf{v}_1$.

4.4.2. (a) $\begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \qquad (c) \begin{pmatrix} \frac{7}{9} \\ \frac{11}{9} \\ \frac{1}{9} \end{pmatrix} \approx \begin{pmatrix} .7778 \\ 1.2222 \\ .1111 \end{pmatrix}.$

4.4.5. Orthogonal basis: $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{3}{2} \\ 2 \\ -\frac{5}{2} \end{pmatrix};$

orthogonal projection: $\dfrac{2}{3}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \dfrac{4}{5}\begin{pmatrix} \frac{3}{2} \\ 2 \\ -\frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{8}{15} \\ \frac{44}{15} \\ -\frac{4}{3} \end{pmatrix} \approx \begin{pmatrix} .5333 \\ 2.9333 \\ -1.3333 \end{pmatrix}.$

4.4.7. (i) (a) $\begin{pmatrix} -\frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix}, \quad (c) \begin{pmatrix} \frac{15}{17} \\ \frac{19}{17} \\ \frac{1}{17} \end{pmatrix} \approx \begin{pmatrix} .88235 \\ 1.11765 \\ .05882 \end{pmatrix}.$

(ii) (a) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (c) \begin{pmatrix} \frac{23}{43} \\ \frac{19}{43} \\ -\frac{1}{43} \end{pmatrix} \approx \begin{pmatrix} .5349 \\ .4419 \\ -.0233 \end{pmatrix}.$

♡ 4.4.9. (a) The entries of $\mathbf{c} = A^T \mathbf{v}$ are $c_i = \mathbf{u}_i^T \mathbf{v} = \mathbf{u}_i \cdot \mathbf{v}$, and hence, by (1.11), $\mathbf{w} = P\mathbf{v} = A\mathbf{c} = c_1 \mathbf{u}_1 + \cdots + c_k \mathbf{u}_k$, reproducing the projection formula (4.41).

(b) (ii) $\begin{pmatrix} \frac{4}{9} & -\frac{4}{9} & \frac{2}{9} \\ -\frac{4}{9} & \frac{4}{9} & -\frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix}$, (iv) $\begin{pmatrix} \frac{1}{9} & -\frac{2}{9} & \frac{2}{9} & 0 \\ -\frac{2}{9} & \frac{8}{9} & 0 & -\frac{2}{9} \\ \frac{2}{9} & 0 & \frac{8}{9} & -\frac{2}{9} \\ 0 & -\frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix}$.

(c) $P^T = (A A^T)^T = A A^T = P$.

4.4.12. (a) $W^\perp$ has basis $\begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}$, $\dim W^\perp = 2$;

(d) $W^\perp$ has basis $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $\dim W^\perp = 1$.

4.4.13. (a) $\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$; (b) $\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$.

4.4.15. (a) $\mathbf{w} = \begin{pmatrix} \frac{3}{10} \\ -\frac{1}{10} \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} \frac{7}{10} \\ \frac{21}{10} \end{pmatrix}$; (c) $\mathbf{w} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$.

4.4.16. (a) Span of $\begin{pmatrix} \frac{2}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$; $\dim W^\perp = 2$. (d) Span of $\begin{pmatrix} 6 \\ \frac{3}{2} \\ 1 \end{pmatrix}$; $\dim W^\perp = 1$.

4.4.19. (a) $\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx = 0$ for all $q(x) = a + bx + cx^2$, or, equivalently, $\int_{-1}^{1} p(x)\, dx = \int_{-1}^{1} x\, p(x)\, dx = \int_{-1}^{1} x^2\, p(x)\, dx = 0$. Writing $p(x) = a + bx + cx^2 + dx^3 + ex^4$, the orthogonality conditions require $2a + \frac{2}{3}c + \frac{2}{5}e = 0$, $\frac{2}{3}b + \frac{2}{5}d = 0$, $\frac{2}{3}a + \frac{2}{5}c + \frac{2}{7}e = 0$.
(b) Basis: $t^3 - \frac{3}{5}t$, $t^4 - \frac{6}{7}t^2 + \frac{3}{35}$; $\dim W^\perp = 2$; (c) the preceding basis is orthogonal.

4.4.22. If $\mathbf{z} \in W_2^\perp$ then $\langle \mathbf{z}, \mathbf{w} \rangle = 0$ for every $\mathbf{w} \in W_2$. In particular, every $\mathbf{w} \in W_1 \subset W_2$, and hence $\mathbf{z}$ is orthogonal to every vector $\mathbf{w} \in W_1$. Thus, $\mathbf{z} \in W_1^\perp$, proving $W_2^\perp \subset W_1^\perp$.

◇ 4.4.25. Suppose $\mathbf{v} \in (W^\perp)^\perp$. Then we write $\mathbf{v} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w} \in W, \mathbf{z} \in W^\perp$. By assumption, for every $\mathbf{y} \in W^\perp$, we must have $0 = \langle \mathbf{v}, \mathbf{y} \rangle = \langle \mathbf{w}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle = \langle \mathbf{z}, \mathbf{y} \rangle$. In particular, when $\mathbf{y} = \mathbf{z}$, this implies $\|\mathbf{z}\|^2 = 0$ and hence $\mathbf{z} = \mathbf{0}$ which proves $\mathbf{v} = \mathbf{w} \in W$.

4.4.29. *Note*: To show orthogonality of two subspaces, it suffices to check orthogonality of their respective basis vectors.

(*a*) (*i*) Image: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; cokernel: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$; coimage: $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$; kernel: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$;

(*ii*) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 0$; (*iii*) $\begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$.

(*c*) (*i*) Image: $\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$; cokernel: $\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$; coimage: $\begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$; kernel: $\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$;

(*ii*) $\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = 0$; (*iii*) $\begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = 0$.

4.4.30. (*a*) The compatibility condition is $\frac{2}{3} b_1 + b_2 = 0$ and so the cokernel basis is $\left(\frac{2}{3}, 1 \right)^T$.

(*c*) There are no compatibility conditions, and so the cokernel is $\{\mathbf{0}\}$.

4.4.32. (*a*) Cokernel basis: $(1, -1, 1)^T$; compatibility condition: $2a - b + c = 0$;

(*c*) cokernel basis: $(-3, 1, 1, 0)^T, (2, -5, 0, 1)^T$;

compatibility conditions: $-3b_1 + b_2 + b_3 = 2b_1 - 5b_2 + b_4 = 0$.

4.4.33. (*a*) $\mathbf{z} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = -\frac{3}{2} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$;

(*c*) $\mathbf{z} = \begin{pmatrix} \frac{14}{17} \\ -\frac{1}{17} \\ -\frac{4}{17} \\ -\frac{5}{17} \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} \frac{3}{17} \\ \frac{1}{17} \\ \frac{4}{17} \\ \frac{5}{17} \end{pmatrix} = \frac{1}{51} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 3 \end{pmatrix} + \frac{4}{51} \begin{pmatrix} 2 \\ 1 \\ 3 \\ 3 \end{pmatrix}$.

4.4.34. (*a*) (*i*) Fredholm requires that the cokernel basis $\left(\frac{1}{2}, 1 \right)^T$ be orthogonal to the right-hand side $(-6, 3)^T$; (*ii*) the general solution is $x = -3 + 2y$ with y free; (*iii*) the minimum norm solution is $x = -\frac{3}{5}$, $y = \frac{6}{5}$.

(*c*) (*i*) Fredholm requires that the cokernel basis $(-1, 3)^T$ be orthogonal to the right-hand side $(12, 4)^T$ (*ii*) the general solution is $x = 2 + \frac{1}{2} y - \frac{3}{2} z$ with y, z free; (*iii*) the minimum norm solution is $x = \frac{4}{7}$, $y = -\frac{2}{7}$, $z = \frac{6}{7}$.

4.4.35. If A is symmetric, $\ker A = \ker A^T = \operatorname{coker} A$, and so this is an immediate consequence of Theorem 4.46.

4.5.1. (a) $t^3 = q_3(t) + \frac{3}{5}\,q_1(t)$, where

$$1 = \frac{\langle\, t^3\,,q_3\,\rangle}{\|\,q_3\,\|^2} = \frac{175}{8}\int_{-1}^{1} t^3\,\left(t^3 - \tfrac{3}{5}\,t\right)dt, \qquad 0 = \frac{\langle\, t^3\,,q_2\,\rangle}{\|\,q_2\,\|^2} = \frac{45}{8}\int_{-1}^{1} t^3\,\left(t^2 - \tfrac{1}{3}\right)dt,$$

$$\frac{3}{5} = \frac{\langle\, t^3\,,q_1\,\rangle}{\|\,q_1\,\|^2} = \frac{3}{2}\int_{-1}^{1} t^3\,t\,dt, \qquad 0 = \frac{\langle\, t^3\,,q_0\,\rangle}{\|\,q_0\,\|^2} = \frac{1}{2}\int_{-1}^{1} t^3\,dt.$$

4.5.2. (a) $q_5(t) = t^5 - \frac{10}{9}\,t^3 + \frac{5}{21}\,t = \dfrac{5\,!}{10\,!}\,\dfrac{d^5}{dt^5}\,(t^2-1)^5$, (b) $t^5 = q_5(t) + \frac{10}{9}\,q_3(t) + \frac{3}{7}\,q_1(t)$.

4.5.6. $q_k(t) = \dfrac{k\,!}{(2\,k)\,!}\,\dfrac{d^k}{dt^k}\,(t^2-1)^k, \qquad \|\,q_k\,\| = \dfrac{2^k\,(k\,!)^2}{(2\,k)\,!}\sqrt{\dfrac{2}{2\,k+1}}\,.$

♡ 4.5.10. (a) The roots of $P_2(t)$ are $\pm\frac{1}{\sqrt{3}}$; the roots of $P_3(t)$ are $0,\ \pm\sqrt{\frac{3}{5}}$.

4.5.11. (a) $P_0(t) = 1$, $P_1(t) = t - \frac{3}{2}$, $P_2(t) = t^2 - 3\,t + \frac{13}{6}$, $P_3(t) = t^3 - \frac{9}{2}\,t^2 + \frac{33}{5}\,t - \frac{63}{20}$.

4.5.15. $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = t^2 - \frac{1}{3}$, $p_3(t) = t^3 - \frac{9}{10}\,t$.

4.5.17. $L_4(t) = t^4 - 16\,t^3 + 72\,t^2 - 96\,t + 24, \qquad \|\,L_4\,\| = 24.$

4.5.22. A basis for the solution set is given by e^x and $e^{2\,x}$. The Gram-Schmidt process yields the orthogonal basis $e^{2\,x}$ and $\dfrac{2\,(e^3-1)}{3\,(e^2-1)}\,e^x$.

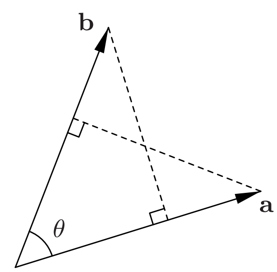
Students' Solutions Manual for
Chapter 5: Minimization and Least Squares

5.1.1. We need to minimize $(3x-1)^2+(2x+1)^2 = 13x^2-2x+2$.
The minimum value of $\frac{25}{13}$ occurs when $x=\frac{1}{13}$.

5.1.3. (b) $(0,2)^T$, (d) $\left(-\frac{3}{2},\frac{3}{2}\right)^T$.

5.1.4. *Note*: To minimize the distance between the point $(a,b)^T$ to the line $y = mx + c$:
(i) in the ∞ norm we must minimize the scalar function $f(x) = \max\big\{\,|x-a\,|,\,|\,mx+c-b\,|\,\big\}$,
while (ii) in the 1 norm we must minimize the scalar function $f(x) = |\,x-a\,|+|\,mx+c-b\,|$.
(i) (b) all points on the line segment $(0,y)^T$ for $1\le y\le 3$; (d) $\left(-\frac{3}{2},\frac{3}{2}\right)^T$.
(ii) (b) $(0,2)^T$; (d) all points on the line segment $(t,-t)^T$ for $-2\le t\le -1$.

5.1.7. This holds because the two triangles in the figure are congruent. According to Exercise 5.1.6(c), when $\|\,\mathbf{a}\,\| = \|\,\mathbf{b}\,\| = 1$, the distance is $|\sin\theta\,|$ where θ is the angle between $\mathbf{a},\mathbf{b}$, as ilustrated:



♡ 5.1.9. (a) The distance is given by $\dfrac{|\,a x_0 + b y_0 + c z_0 + d\,|}{\sqrt{a^2+b^2+c^2}}$. (b) $\dfrac{1}{\sqrt{14}}$.

5.2.1. $x=\frac{1}{2}$, $y=\frac{1}{2}$, $z=-2$, with $f(x,y,z)=-\frac{3}{2}$. This is the global minimum because the coefficient matrix $\begin{pmatrix}1&1&0\\1&3&1\\0&1&1\end{pmatrix}$ is positive definite.

5.2.3. (b) Minimizer: $x=\frac{2}{9}$, $y=\frac{2}{9}$; minimum value: $\frac{32}{9}$.
(d) Minimizer: $x=-\frac{1}{2}$, $y=-1$, $z=1$; minimum value: $-\frac{5}{4}$.
(f) No minimum.

5.2.5. (a) $p(\mathbf{x}) = 4x^2-24xy+45y^2+x-4y+3$; minimizer: $\mathbf{x}^\star = \left(\frac{1}{24},\frac{1}{18}\right)^T \approx (.0417, .0556)^T$; minimum value: $p(\mathbf{x}^\star) = \frac{419}{144} \approx 2.9097$.
(b) $p(\mathbf{x}) = 3x^2+4xy+y^2-8x-2y$; no minimizer since K is not positive definite.

5.2.7. (a) maximizer: $\mathbf{x}^\star = \left(\frac{10}{11}, \frac{3}{11}\right)^T$; maximum value: $p(\mathbf{x}^\star) = \frac{16}{11}$.

◇ 5.2.9. Let $\mathbf{x}^\star = K^{-1}\mathbf{f}$ be the minimizer. When $c = 0$, according to the third expression in (5.14), $p(\mathbf{x}^\star) = -(\mathbf{x}^\star)^T K \mathbf{x}^\star \leq 0$ because K is positive definite. The minimum value is 0 if and only if $\mathbf{x}^\star = \mathbf{0}$, which occurs if and only if $\mathbf{f} = \mathbf{0}$.

5.2.13. False. See Example 4.51 for a counterexample.

5.3.1. Closest point: $\left(\frac{6}{7}, \frac{38}{35}, \frac{36}{35}\right)^T \approx (.85714, 1.08571, 1.02857)^T$; distance: $\frac{1}{\sqrt{35}} \approx .16903$.

5.3.2. (a) Closest point: $(.8343, 1.0497, 1.0221)^T$; distance: .2575.

5.3.4. (a) Closest point: $\left(\frac{7}{4}, \frac{7}{4}, \frac{7}{4}, \frac{7}{4}\right)^T$; distance: $\sqrt{\frac{11}{4}}$.
(c) Closest point: $(3, 1, 2, 0)^T$; distance: 1.

5.3.6. (i) Exercise 5.3.4: (a) Closest point: $\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)^T$; distance: $\sqrt{\frac{7}{4}}$.
(c) Closest point: $(3, 1, 2, 0)^T$; distance: 1.
(ii) Exercise 5.3.4: (a) Closest point: $\left(\frac{25}{14}, \frac{25}{14}, \frac{25}{14}, \frac{25}{14}\right)^T \approx (1.7857, 1.7857, 1.7857, 1.7857)^T$;
distance: $\sqrt{\frac{215}{14}} \approx 3.9188$.
(c) Closest point: $\left(\frac{28}{9}, \frac{11}{9}, \frac{16}{9}, 0\right)^T \approx (3.1111, 1.2222, 1.7778, 0)^T$;
distance: $\sqrt{\frac{32}{9}} \approx 1.8856$.

5.3.8. (a) $\sqrt{\frac{8}{3}}$.

5.3.12. $\left(\frac{1}{2}, -\frac{1}{2}, 2\right)^T$.

5.3.14. Orthogonal basis: $(1, 0, 2, 1)^T, (1, 1, 0, -1)^T, \left(\frac{1}{2}, -1, 0, -\frac{1}{2}\right)^T$;
closest point = orthogonal projection = $\left(-\frac{2}{3}, 2, \frac{2}{3}, \frac{4}{3}\right)^T$.

5.4.1. (a) $\frac{1}{2}$, (b) $\begin{pmatrix} \frac{8}{5} \\ \frac{28}{65} \end{pmatrix} = \begin{pmatrix} 1.6 \\ .4308 \end{pmatrix}$.

5.4.2. (b) $x = -\frac{1}{25}$, $y = -\frac{8}{21}$; (d) $x = \frac{1}{3}$, $y = 2$, $z = \frac{3}{4}$.

5.4.5. The solution is $\mathbf{x}^\star = (-1, 2, 3)^T$. The least squares error is 0 because $\mathbf{b} \in \operatorname{img} A$ and so $\mathbf{x}^\star$ is an exact solution.

5.4.8. The solutions are, of course, the same:

$$(b)\quad Q = \begin{pmatrix} .8 & -.43644 \\ .4 & .65465 \\ .2 & -.43644 \\ .4 & .43644 \end{pmatrix}, \qquad R = \begin{pmatrix} 5 & 0 \\ 0 & 4.58258 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} -.04000 \\ -.38095 \end{pmatrix};$$

$$(d)\ \ Q = \begin{pmatrix} .18257 & .36515 & .12910 \\ .36515 & -.18257 & .90370 \\ 0 & .91287 & .12910 \\ -.91287 & 0 & .38730 \end{pmatrix}, \qquad R = \begin{pmatrix} 5.47723 & -2.19089 & 0 \\ 0 & 1.09545 & -3.65148 \\ 0 & 0 & 2.58199 \end{pmatrix},$$

$$\mathbf{x} = (\, .33333, 2.00000, .75000\,)^T.$$

5.4.10. (a) $\left(-\frac{1}{7}, 0\right)^T$, $\quad$ (b) $\left(\frac{9}{14}, \frac{4}{31}\right)^T$.

5.5.1. (b) $y = 1.9 - 1.1\,t$.

5.5.3. (a) $y = 3.9227\,t - 7717.7$; $\quad$ (b) \$147,359 and \$166,973.

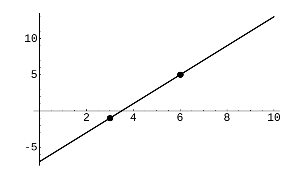
5.5.6. (a) The least squares exponential is $y = e^{4.6051 - .1903\,t}$ and, at $t = 10$, $y = 14.9059$.
(b) Solving $e^{4.6051 - .1903\,t} = .01$, we find $t = 48.3897 \approx 49$ days.

5.5.8. (a) $.29 + .36\,t$, $\quad$ (c) $-1.2308 + 1.9444\,t$.

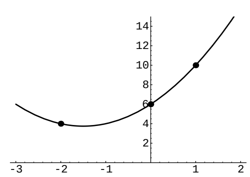
♢ 5.5.12.

$$\frac{1}{m}\sum_{i=1}^{m}(t_i - \overline{t}\,)^2 = \frac{1}{m}\sum_{i=1}^{m} t_i^2 - \frac{2\,\overline{t}}{m}\sum_{i=1}^{m} t_i + \frac{(\,\overline{t}\,)^2}{m}\sum_{i=1}^{m} 1 = \overline{t^2} - 2\,(\,\overline{t}\,)^2 + (\,\overline{t}\,)^2 = \overline{t^2} - (\,\overline{t}\,)^2.$$

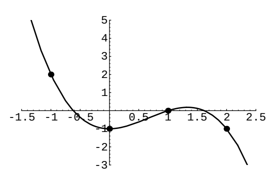
5.5.13. (a) $y = 2\,t - 7$;



(b) $y = t^2 + 3\,t + 6$;



(d) $y = -t^3 + 2t^2 - 1$.

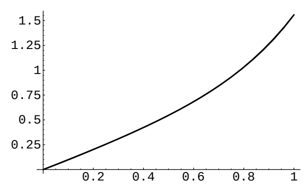


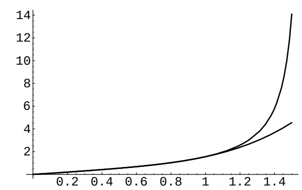
5.5.14. (a) $p(t) = -\frac{1}{5}\,(t-2) + (t+3) = \frac{17}{5} + \frac{4}{5}\,t$,
(b) $p(t) = \frac{1}{3}\,(t-1)(t-3) - \frac{1}{4}\,t(t-3) + \frac{1}{24}\,t(t-1) = 1 - \frac{5}{8}\,t + \frac{1}{8}\,t^2$,
(d) $p(t) = \frac{1}{2}\,t\,(t-2)(t-3) - 2\,t\,(t-1)(t-3) + \frac{3}{2}\,t\,(t-1)(t-2) = t^2$.

5.5.17. The quadratic least squares polynomial is $y = 4480.5 + 6.05\,t - 1.825\,t^2$, and $y = 1500$ at $t = 42.1038$ seconds.

5.5.20. (a) $p_2(t) = 1 + t + \frac{1}{2}t^2$, whose maximal error over $[0,1]$ is .218282.

5.5.22. $p(t) = .9409\,t + .4566\,t^2 - .7732\,t^3 + .9330\,t^4$. The graphs are very close over the interval $0 \le t \le 1$; the maximum error is .005144 at $t = .91916$. The functions rapidly diverge above 1, with $\tan t \to \infty$ as $t \to \frac{1}{2}\pi$, whereas $p(\frac{1}{2}\pi) = 5.2882$. The first graph is on the interval $[0,1]$ and the second on $[0, \frac{1}{2}\pi]$:





5.5.26. (a) $p_1(t) = 14 + \frac{7}{2}t,\ \ p_2(t) = p_3(t) = 14 + \frac{7}{2}t + \frac{1}{14}(t^2 - 2)$.

5.5.28. (a) $p_4(t) = 14 + \frac{7}{2}t + \frac{1}{14}(t^2 - 2) - \frac{5}{12}\left(t^4 - \frac{31}{7} + \frac{72}{35}\right)$.

♢ 5.5.31.

$$q_0(t) = 1, \qquad q_1(t) = t - \overline{t}, \qquad q_2(t) = t^2 - \frac{\overline{t^3} - \overline{t}\,\overline{t^2}}{\overline{t^2} - \overline{t}^2}\left(t - \overline{t}\right) - \overline{t^2},$$

$$\mathbf{q}_0 = \mathbf{t}_0, \qquad \mathbf{q}_1 = \mathbf{t}_1 - \overline{t}\,\mathbf{t}_0, \qquad \mathbf{q}_2 = \mathbf{t}_2 - \frac{\overline{t^3} - \overline{t}\,\overline{t^2}}{\overline{t^2} - \overline{t}^2}\left(\mathbf{t}_1 - \overline{t}\right) - \overline{t^2}\,\mathbf{t}_0,$$

$$\|\mathbf{q}_0\|^2 = 1, \qquad \|\mathbf{q}_1\|^2 = \overline{t^2} - \overline{t}^2, \qquad \|\mathbf{q}_2\|^2 = \overline{t^4} - \left(\overline{t^2}\right)^2 - \frac{\left(\overline{t^3} - \overline{t}\,\overline{t^2}\right)^2}{\overline{t^2} - \overline{t}^2}.$$

5.5.35. (a) For example, an interpolating polynomial for the data $(0,0), (1,1), (2,2)$ is the straight line $y = t$.

♢ 5.5.38. (a) If $p(x_k) = a_0 + a_1 x_k + a_2 x_k^2 + \cdots + a_n x_k^n = 0$ for $k = 1, \ldots, n+1$, then $V\mathbf{a} = \mathbf{0}$ where V is the $(n+1)\times(n+1)$ Vandermonde matrix with entries $v_{ij} = x_j^{i-1}$ for $i,j = 1,\ldots,n+1$. According to Lemma 5.16, if the sample points are distinct, then V is a nonsingular matrix, and hence the only solution to the homogeneous linear system is $\mathbf{a} = \mathbf{0}$, which implies $p(x) \equiv 0$.

♡ 5.5.39. (a) $f'(x) \approx \dfrac{f(x+h) - f(x-h)}{2h}$; (b) $f''(x) \approx \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$

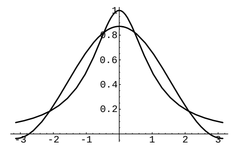
♡ 5.5.40. (a) Trapezoid Rule: $\displaystyle\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\big[\,f(x_0) + f(x_1)\,\big]$.

(b) Simpson's Rule: $\displaystyle\int_a^b f(x)\,dx \approx \frac{1}{6}(b-a)\big[\,f(x_0) + 4f(x_1) + f(x_2)\,\big]$.

5.5.41. The sample matrix is $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$; the least squares solution to $A\,\mathbf{x} = \mathbf{y} = \begin{pmatrix} 1 \\ .5 \\ .25 \end{pmatrix}$ gives $g(t) = \frac{3}{8}\cos\pi t + \frac{1}{2}\sin\pi t$.

5.5.43. (a) $g(t) = .538642\,e^{t} - .004497\,e^{2t}$, (b) .735894.
(c) The maximal error is .745159 which occurs at $t = 3.66351$.

♡ 5.5.45. (a) $n = 1, k = 4$: $p(t) = .4172 + .4540\cos t$;
maximal error: .1722;



5.5.47. (a) $\frac{3}{7} + \frac{9}{14}\,t$; (b) $\frac{9}{28} + \frac{9}{7}\,t - \frac{9}{14}\,t^2$.

♡ 5.5.51. (a) $1.875\,x^2 - .875\,x$, (c) $1.7857\,x^2 - 1.0714\,x + .1071$.

5.5.54. (i) $\frac{3}{28} - \frac{15}{14}\,t + \frac{25}{14}\,t^2 \approx .10714 - 1.07143\,t + 1.78571\,t^2$; maximal error: $\frac{5}{28} = .178571$ at $t = 1$; (ii) $\frac{2}{7} - \frac{25}{14}\,t + \frac{50}{21}\,t^2 \approx .28571 - 1.78571\,t + 2.38095\,t^2$; maximal error: $\frac{2}{7} = .285714$ at $t = 0$.

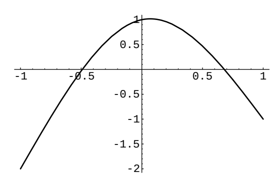
5.5.56. (a) $z = x + y - \frac{1}{3}$, (b) $z = \frac{9}{10}\,(x - y)$.

5.5.58. (a) $\frac{1}{5} + \frac{4}{7}\left(-\frac{1}{2} + \frac{3}{2}\,t^2\right) = -\frac{3}{35} + \frac{6}{7}\,t^2$.

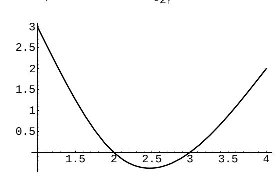
5.5.60. $1.718282 + .845155\,(2t-1) + .139864\,(6t^2 - 6t + 1) + .013931\,(20t^3 - 30t^2 + 12t - 1)$
$= .99906 + 1.0183\,t + .421246\,t^2 + .278625\,t^3$.

♠ 5.5.63. $.459698 + .427919\,(2t-1) - .0392436\,(6t^2 - 6t + 1) - .00721219\,(20t^3 - 30t^2 + 12t - 1)$
$= -.000252739 + 1.00475\,t - .0190961\,t^2 - .144244\,t^3$.

5.5.67. (a) $u(x) = \begin{cases} -1.25\,(x+1)^3 + 4.25\,(x+1) - 2, & -1 \le x \le 0, \\ 1.25\,x^3 - 3.75\,x^2 + .5\,x - 1, & 0 \le x \le 1. \end{cases}$

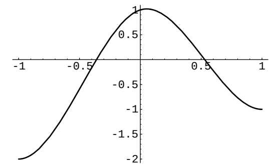


(c) $u(x) = \begin{cases} \frac{2}{3}\,(x-1)^3 - \frac{11}{3}\,(x-1) + 3, & 1 \le x \le 2, \\ -\frac{1}{3}\,(x-2)^3 + 2\,(x-2)^2 - \frac{5}{3}\,(x-2), & 2 \le x \le 4. \end{cases}$

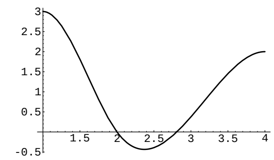


5.5.68.

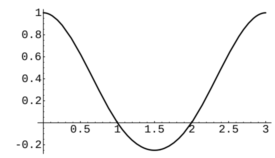
(a) $u(x) = \begin{cases} -5.25\,(x+1)^3 + 8.25\,(x+1)^2 - 2, & -1 \le x \le 0, \\ 4.75\,x^3 - 7.5\,x^2 + .75\,x + 1, & 0 \le x \le 1. \end{cases}$



(c) $u(x) = \begin{cases} 3.5\,(x-1)^3 - 6.5\,(x-1)^2 + 3, & 1 \le x \le 2, \\ -1.125\,(x-2)^3 + 4\,(x-2)^2 - 2.5\,(x-2), & 2 \le x \le 4. \end{cases}$

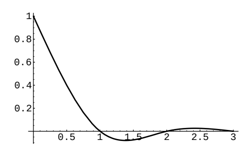


5.5.69. (a) $u(x) = \begin{cases} x^3 - 2\,x^2 + 1, & 0 \le x \le 1, \\ (x-1)^2 - (x-1), & 1 \le x \le 2, \\ -(x-2)^3 + (x-2)^2 + (x-2), & 2 \le x \le 3. \end{cases}$

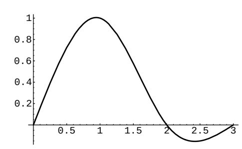


♡ 5.5.75. (a)

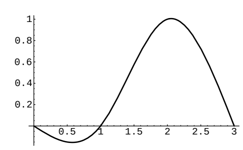
$$C_0(x) = \begin{cases} 1 - \frac{19}{15}\,x + \frac{4}{15}\,x^3, & 0 \le x \le 1, \\ -\frac{7}{15}\,(x-1) + \frac{4}{5}(x-1)^2 - \frac{1}{3}\,(x-1)^3, & 1 \le x \le 2, \\ \frac{2}{15}\,(x-2) - \frac{1}{5}\,(x-2)^2 + \frac{1}{15}\,(x-2)^3, & 2 \le x \le 3, \end{cases}$$



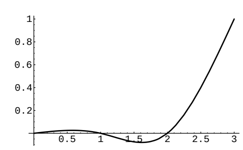
$$C_1(x) = \begin{cases} \frac{8}{5}\,x - \frac{3}{5}\,x^3, & 0 \le x \le 1, \\ 1 - \frac{1}{5}\,(x-1) - \frac{9}{5}(x-1)^2 + (x-1)^3, & 1 \le x \le 2, \\ -\frac{4}{5}\,(x-2) + \frac{6}{5}\,(x-2)^2 - \frac{2}{5}\,(x-2)^3, & 2 \le x \le 3, \end{cases}$$



$$C_2(x) = \begin{cases} -\frac{2}{5}\,x + \frac{2}{5}\,x^3, & 0 \le x \le 1, \\ \frac{4}{5}\,(x-1) + \frac{6}{5}(x-1)^2 - (x-1)^3, & 1 \le x \le 2, \\ \frac{1}{5}\,(x-2) - \frac{9}{5}\,(x-2)^2 + \frac{3}{5}\,(x-2)^3, & 2 \le x \le 3, \end{cases}$$



$$C_3(x) = \begin{cases} \frac{1}{15}\,x - \frac{1}{15}\,x^3, & 0 \le x \le 1, \\ -\frac{1}{2}15\,(x-1) - \frac{1}{5}(x-1)^2 + \frac{1}{3}\,(x-1)^3, & 1 \le x \le 2, \\ \frac{7}{15}\,(x-2) + \frac{4}{5}\,(x-2)^2 - \frac{4}{15}\,(x-2)^3, & 2 \le x \le 3. \end{cases}$$



(*b*) It suffices to note that any linear combination of natural splines is a natural spline. Moreover, $u(x_j) = y_0\,C_0(x_j) + y_1\,C_1(x_j) + \cdots + y_n\,C_n(x_j) = y_j$, as desired.

5.6.1. (a) (*i*) $c_0 = 0,\ c_1 = -\frac{1}{2}\,\mathrm{i}\,,\ c_2 = c_{-2} = 0,\ c_3 = c_{-1} = \frac{1}{2}\,\mathrm{i}\,,$ (*ii*) $\frac{1}{2}\,\mathrm{i}\,e^{-\,\mathrm{i}\,x} - \frac{1}{2}\,\mathrm{i}\,e^{\,\mathrm{i}\,x} = \sin x;$

(*c*) (*i*) $c_0 = \frac{1}{3},\ c_1 = \frac{3-\sqrt{3}\,\mathrm{i}}{12},\ c_2 = \frac{1-\sqrt{3}\,\mathrm{i}}{12},\ c_3 = c_{-3} = 0,\ c_4 = c_{-2} = \frac{1+\sqrt{3}\,\mathrm{i}}{12},$

$$c_5 = c_{-1} = \frac{3+\sqrt{3}\,\mathrm{i}}{12},\ (ii)\ \frac{1+\sqrt{3}\,\mathrm{i}}{12}\,e^{-2\,\mathrm{i}\,x} + \frac{3+\sqrt{3}\,\mathrm{i}}{12}\,e^{-\,\mathrm{i}\,x} + \frac{1}{3} + \frac{3-\sqrt{3}\,\mathrm{i}}{12}\,e^{\,\mathrm{i}\,x} + \frac{1-\sqrt{3}\,\mathrm{i}}{12}\,e^{2\,\mathrm{i}\,x}$$
$$= \frac{1}{3} + \frac{1}{2}\cos x + \frac{1}{2\sqrt{3}}\sin x + \frac{1}{6}\cos 2x + \frac{1}{2\sqrt{3}}\sin 2x.$$

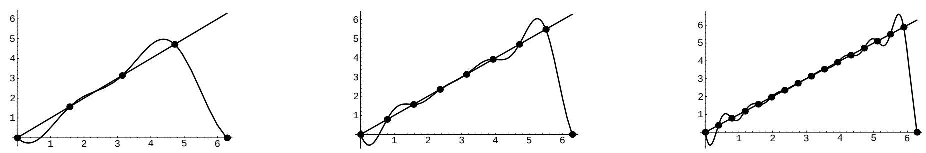
5.6.2. (a) (*i*) $f_0 = 2,\ f_1 = -1,\ f_2 = -1.$ (*ii*) $e^{-\,\mathrm{i}\,x} + e^{\,\mathrm{i}\,x} = 2\cos x;$

(*c*) (*i*) $f_0 = 6,\ f_1 = 2 + 2e^{2\pi\,\mathrm{i}/5} + 2e^{-4\pi\,\mathrm{i}/5} = 1 + .7265\,\mathrm{i}\,,$

$$f_2 = 2 + 2e^{2\pi\,\mathrm{i}/5} + 2e^{4\pi\,\mathrm{i}/5} = 1 + 3.0777\,\mathrm{i}\,,$$
$$f_3 = 2 + 2e^{-2\pi\,\mathrm{i}/5} + 2e^{-4\pi\,\mathrm{i}/5} = 1 - 3.0777\,\mathrm{i}\,,$$
$$f_4 = 2 + 2e^{-2\pi\,\mathrm{i}/5} + 2e^{4\pi\,\mathrm{i}/5} = 1 - .7265\,\mathrm{i}\,;$$

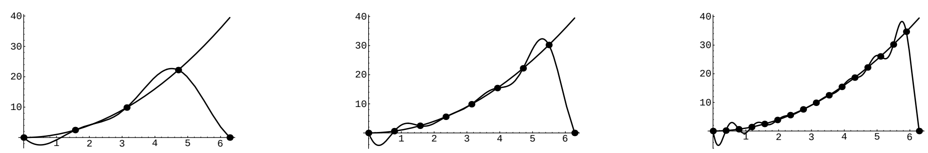
(*ii*) $2e^{-2\,\mathrm{i}\,x} + 2 + 2e^{\,\mathrm{i}\,x} = 2 + 2\cos x + 2\,\mathrm{i}\,\sin x + 2\cos 2x - 2\,\mathrm{i}\,\sin 2x.$

♠ 5.6.3.

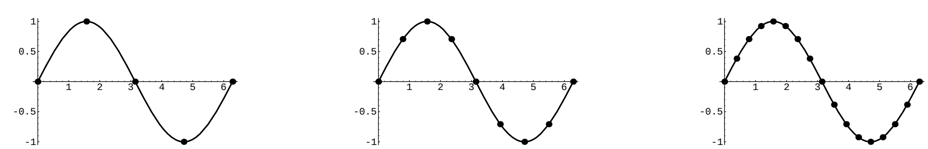


The interpolants are accurate along most of the interval, but there is a noticeable problem near the endpoints $x = 0, 2\pi$. (In Fourier theory, [**19**, **61**], this is known as the Gibbs phenomenon.)

♠ 5.6.4. (a)

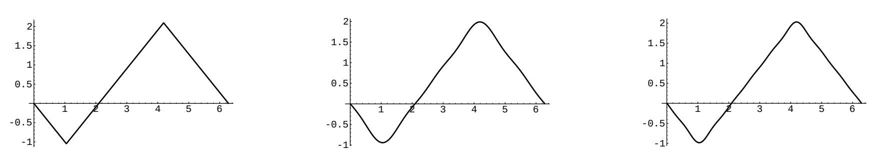


(*c*)



♢ 5.6.7. (a) (*i*) $\mathrm{i}\,, -\mathrm{i}\,;$ (*ii*) $e^{2\pi k\,\mathrm{i}/5}$ for $k = 1, 2, 3$ or 4; (*iii*) $e^{2\pi k\,\mathrm{i}/9}$ for $k = 1, 2, 4, 5, 7$ or 8.

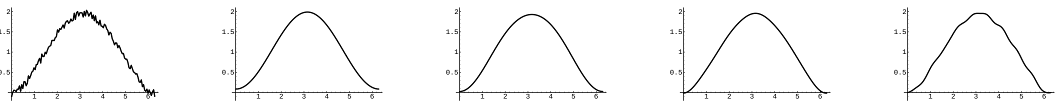
♠ 5.6.10.



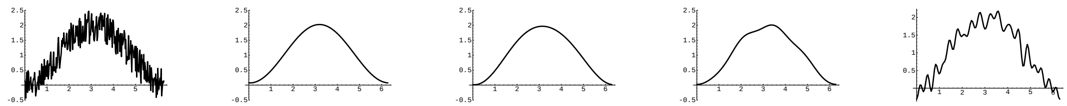
Original function, 11 mode compression, 21 mode compression.

The average absolute errors are .018565 and .007981; the maximal errors are .08956 and .04836, so the 21 mode compression is about twice as accurate.

♣ 5.6.13. Very few are needed. In fact, if you take too many modes, you do worse! For example, if $\varepsilon = .1$,



plots the noisy signal and the effect of retaining $2l+1 = 3, 5, 11, 21$ modes. Only the first three give reasonable results. When $\varepsilon = .5$ the effect is even more pronounced:



♠ 5.6.17. (a) $\mathbf{f} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \\ \frac{3}{2} \end{pmatrix}, \ c^{(0)} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}, \ c^{(1)} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}, \ c = c^{(2)} = \begin{pmatrix} \frac{3}{4} \\ -\frac{1}{4} + \frac{1}{4}\,\mathrm{i} \\ -\frac{1}{4} \\ -\frac{1}{4} + \frac{1}{4}\,\mathrm{i} \end{pmatrix};$

$$(c)\ \ \mathbf{f} = \begin{pmatrix} \pi \\ \frac{3}{4}\pi \\ \frac{1}{2}\pi \\ \frac{1}{4}\pi \\ 0 \\ \frac{1}{4}\pi \\ \frac{1}{2}\pi \\ \frac{3}{4}\pi \end{pmatrix}, \ c^{(0)} = \begin{pmatrix} \pi \\ 0 \\ \frac{1}{2}\pi \\ \frac{1}{2}\pi \\ \frac{3}{4}\pi \\ \frac{1}{4}\pi \\ \frac{1}{4}\pi \\ \frac{3}{4}\pi \end{pmatrix}, \ c^{(1)} = \begin{pmatrix} \frac{1}{2}\pi \\ \frac{1}{2}\pi \\ \frac{1}{2}\pi \\ 0 \\ \frac{1}{2}\pi \\ \frac{1}{4}\pi \\ \frac{1}{2}\pi \\ -\frac{1}{4}\pi \end{pmatrix}, \ c^{(2)} = \begin{pmatrix} \frac{1}{2}\pi \\ \frac{1}{4}\pi \\ 0 \\ \frac{1}{4}\pi \\ \frac{1}{2}\pi \\ \frac{1+\mathrm{i}}{8}\pi \\ 0 \\ \frac{1-\mathrm{i}}{8}\pi \end{pmatrix}, \ c = c^{(3)} = \begin{pmatrix} \frac{1}{2}\pi \\ \frac{\sqrt{2}+1}{8\sqrt{2}}\pi \\ 0 \\ \frac{\sqrt{2}-1}{8\sqrt{2}}\pi \\ 0 \\ \frac{\sqrt{2}-1}{8\sqrt{2}}\pi \\ 0 \\ \frac{\sqrt{2}+1}{8\sqrt{2}}\pi \end{pmatrix}.$$

♠ 5.6.18. (a) $\mathbf{c} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{f}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{f}^{(1)} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \quad \mathbf{f} = \mathbf{f}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 0 \end{pmatrix}$

Students' Solutions Manual for
Chapter 6: Equilibrium

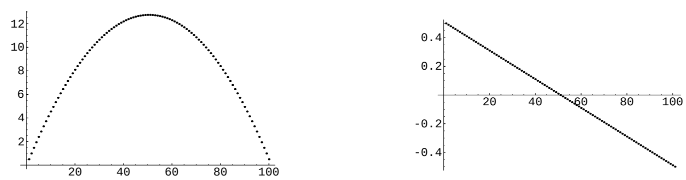
6.1.1. (a) $K = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$; (b) $\mathbf{u} = \begin{pmatrix} \frac{18}{5} \\ \frac{17}{5} \end{pmatrix} = \begin{pmatrix} 3.6 \\ 3.4 \end{pmatrix}$; (c) the first mass has moved the farthest; (d) $\mathbf{e} = \left(\frac{18}{5}, -\frac{1}{5}, -\frac{17}{5} \right)^T = (\,3.6, -.2, -3.4\,)^T$, so the first spring has stretched the most, while the third spring experiences the most compression.

6.1.3. Exercise 6.1.1: (a) $K = \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}$; (b) $\mathbf{u} = \begin{pmatrix} 7 \\ \frac{17}{2} \end{pmatrix} = \begin{pmatrix} 7.0 \\ 8.5 \end{pmatrix}$;

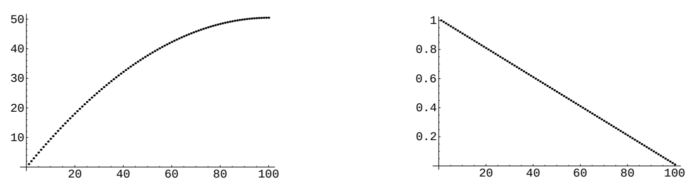
(c) the second mass has moved the farthest;

(d) $\mathbf{e} = \left(7, \frac{3}{2} \right)^T = (\,7.0, 1.5\,)^T$, so the first spring has stretched the most.

♣ 6.1.7. Top and bottom support; constant force:



Top support only; constant force:



6.1.8. (a) For maximum displacement of the bottom mass, the springs should be arranged from weakest at the top to strongest at the bottom, so $c_1 = c = 1$, $c_2 = c' = 2$, $c_3 = c'' = 3$.

6.1.13. Denoting the gravitation force by g:

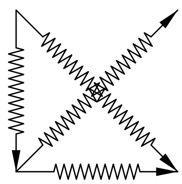
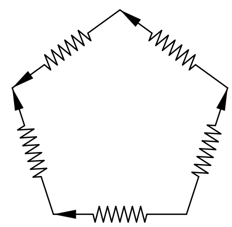
(a) $$p(\mathbf{u}) = \frac{1}{2} \,(\, u_1 \; u_2 \; u_3 \,) \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} - (\, u_1 \; u_2 \; u_3 \,) \begin{pmatrix} g \\ g \\ g \end{pmatrix}$$
$$= u_1^2 - u_1 u_2 + u_2^2 - u_2 u_3 + \tfrac{1}{2} u_3^2 - g\,(u_1 + u_2 + u_3).$$

6.1.14. (a) $$p(\mathbf{u}) = \frac{1}{2} \,(\, u_1 \; u_2 \,) \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - (\, u_1 \; u_2 \,) \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$
$$= \tfrac{3}{2} u_1^2 - 2 u_1 u_2 + \tfrac{3}{2} u_2^2 - 4 u_1 - 3 u_2, \quad \text{so } p(\mathbf{u}^\star) = p\big(\,3.6, 3.4\,\big) = -12.3.$$

(b) For instance, $p(1,0) = -2.5$, $p(0,1) = -1.5$, $p(3,3) = -12$.

6.1.16. (a) Two masses, both ends fixed, $c_1 = 2$, $c_2 = 4$, $c_3 = 2$, $\mathbf{f} = (-1, 3)^T$;
equilibrium: $\mathbf{u}^\star = (.3, .7)^T$.

(c) Three masses, top end fixed, $c_1 = 1$, $c_2 = 3$, $c_3 = 5$, $\mathbf{f} = (1, 1, -1)^T$;
equilibrium: $\mathbf{u}^\star = \left(1, 1, \frac{4}{5}\right)^T = (1, 1, .8)^T$.

6.2.1. (b)  (d) 

6.2.2. (a) $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$; (b) $\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$.

(c) $\mathbf{u} = \left(\frac{15}{8},\ \frac{9}{8},\ \frac{3}{2}\right)^T = (1.875,\ 1.125,\ 1.5)^T$;

(d) The currents are

$$\mathbf{y} = \mathbf{v} = A\,\mathbf{u} = \left(\frac{3}{4},\ \frac{3}{8},\ \frac{15}{8},\ -\frac{3}{8},\ \frac{9}{8}\right)^T = (.75, .375, 1.875, -.375, 1.125)^T,$$

and hence the bulb will be brightest when connected to wire 3, which has the most current flowing through it.

♠ 6.2.6. None.

♠ 6.2.8. (a) The potentials remain the same, but the currents are all twice as large.

6.2.12. (a) True, since they satisfy the same systems of equilibrium equations $K\mathbf{u} = -A^T C\,\mathbf{b} = \mathbf{f}$.

(b) False, because the currents with the batteries are, by (6.37), $\mathbf{y} = C\,\mathbf{v} = C\,A\,\mathbf{u} + C\,\mathbf{b}$, while for the current sources they are $\mathbf{y} = C\,\mathbf{v} = C\,A\,\mathbf{u}$.

6.2.13. (a) (i) $\mathbf{u} = (2, 1, 1, 0)^T$, $\mathbf{y} = (1, 0, 1)^T$; (iii) $\mathbf{u} = (3, 2, 1, 1, 1, 0)^T$, $\mathbf{y} = (1, 1, 0, 0, 1)^T$.

6.2.14. (i) $\mathbf{u} = \left(\frac{3}{2}, \frac{1}{2}, 0, 0\right)^T$, $\mathbf{y} = \left(1, \frac{1}{2}, \frac{1}{2}\right)^T$; (iii) $\mathbf{u} = \left(\frac{7}{3}, \frac{4}{3}, \frac{1}{3}, 0, 0, 0\right)^T$, $\mathbf{y} = \left(1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$.

6.2.17. (a) If $\mathbf{f}$ are the current sources at the nodes and $\mathbf{b}$ the battery terms, then the nodal voltage potentials satisfy $A^T C\,A\,\mathbf{u} = \mathbf{f} - A^T C\,\mathbf{b}$.

(b) By linearity, the combined potentials (currents) are obtained by adding the potentials (currents) due to the batteries and those resulting from the current sources.

(c) $\frac{1}{2}P = p(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T K\,\mathbf{u} - \mathbf{u}^T(\mathbf{f} - A^T C\,\mathbf{b})$.

6.3.1. 8 cm

6.3.3. (a) For a unit horizontal force on the two nodes, the displacement vector is $\mathbf{u} = (\,1.5, -.5, 2.5, 2.5\,)^T$, so the left node has moved slightly down and three times as far to the right, while the right node has moved five times as far up and to the right. Note that the force on the left node is transmitted through the top bar to the right node, which explains why it moves significantly further. The stresses are $\mathbf{e} = (\,.7071, 1, 0, -1.5811\,)^T$, so the left and the top bar are elongated, the right bar is stress-free, and the reinforcing bar is significantly compressed.

♡ 6.3.5. (a) $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$; (b) $$\begin{aligned} \tfrac{3}{2}u_1 - \tfrac{1}{2}v_1 - u_2 &= f_1, \\ -\tfrac{1}{2}u_1 + \tfrac{3}{2}v_1 &= g_1, \\ -u_1 + \tfrac{3}{2}u_2 + \tfrac{1}{2}v_2 &= f_2, \\ \tfrac{1}{2}u_2 + \tfrac{3}{2}v_2 &= g_2. \end{aligned}$$

(c) Stable, statically indeterminate.

(d) Write down $\mathbf{f} = K\,\mathbf{e}_1$, so $\mathbf{f}_1 = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$, $\mathbf{f}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. The horizontal bar is under the most stress: it is compressed by -1; the upper left to lower right bar is compressed $-\frac{1}{\sqrt{2}}$, while all other bars are stress free.

♡ 6.3.8. (a) $$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ $$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -.9487 & -.3162 & .9487 & .3162 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -.9487 & .3162 & .9487 & -.3162 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

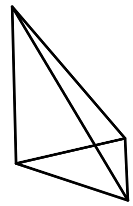
(b) One instability: the mechanism of simultaneous horizontal motion of the three nodes.

(c) No net horizontal force: $f_1 + f_2 + f_3 = 0$. For example, if $\mathbf{f}_1 = \mathbf{f}_2 = \mathbf{f}_3 = (\,0, 1\,)^T$, then $\mathbf{e} = \left(\frac{3}{2}, \sqrt{\frac{5}{2}}, -\frac{3}{2}, \sqrt{\frac{5}{2}}, \frac{3}{2}\right)^T = (\,1.5, 1.5811, -1.5, 1.5811, 1.5\,)^T$, so the compressed diagonal bars have slightly more stress than the compressed vertical bars or the elongated horizontal bar.

(d) To stabilize, add in one more bar starting at one of the fixed nodes and going to one of the two movable nodes not already connected to it.

(e) In every case, $\mathbf{e} = \left(\frac{3}{2}, \sqrt{\frac{5}{2}}, -\frac{3}{2}, \sqrt{\frac{5}{2}}, \frac{3}{2}, 0\right)^T = (\,1.5, 1.5811, -1.5, 1.5811, 1.5, 0\,)^T$, so the stresses on the previous bars are all the same, while the reinforcing bar experiences no stress. (See Exercise 6.3.19 for the general principle.)

♣ 6.3.11. (a)



$$A = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix};$$

$$(b)\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

(c) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ correspond to translations in, respectively, the x, y, z directions;

(d) $\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$ correspond to rotations around, respectively, the x, y, z coordinate axes;

$$(e)\ K = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 2 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 2 \end{pmatrix};$$

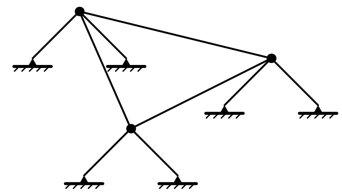
(f) For $\mathbf{f}_i = (f_i, g_i, h_i)^T$ we require $f_1 + f_2 + f_3 + f_4 = 0$, $g_1 + g_2 + g_3 + g_4 = 0$, $h_1 + h_2 + h_3 + h_4 = 0$, $h_3 = g_4$, $h_2 = f_4$, $g_2 = f_3$, i.e., there is no net horizontal force and no net moment of force around any axis.

(g) You need to fix three nodes. Fixing two still leaves a rotation motion around the line connecting them.

(h) Displacement of the top node: $\mathbf{u}_4 = (-1, -1, -1)^T$; since $\mathbf{e} = (-1, 0, 0, 0)^T$, only the vertical bar experiences compression of magnitude 1.

6.3.14. (a) $3n$.

(b) Example: a triangle each of whose nodes is connected to the ground by two additional, non-parallel bars.



◊ 6.3.18. (a) We are assuming that $\mathbf{f} \in \operatorname{img} K = \operatorname{coimg} A = \operatorname{img} A^T$, cf. Exercise 3.4.32. Thus, we can write $\mathbf{f} = A^T\mathbf{h} = A^T C\,\mathbf{g}$ where $\mathbf{g} = C^{-1}\mathbf{h}$.

(b) The equilibrium equations $K\mathbf{u} = \mathbf{f}$ are $A^T C A\,\mathbf{u} = A^T C\,\mathbf{g}$ which are the normal equations (5.36) for the weighted least squares solution to $A\,\mathbf{u} = \mathbf{g}$.

♡ 6.3.20. (a) $A^\star = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$;

$$K^\star\,\mathbf{u} = \mathbf{f}^\star \text{ where } K^\star = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -1 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(b) Unstable, since there are two mechanisms prescribed by the kernel basis elements $(1, -1, 1, 1, 0)^T$, which represents the same mechanism as when the end is fixed, and $(1, -1, 1, 0, 1)^T$, in which the roller and the right hand node move horizontally to the right, while the left node moves down and to the right.

6.3.24. (a) True. Since $K\mathbf{u} = \mathbf{f}$, if $\mathbf{f} \neq \mathbf{0}$ then $\mathbf{u} \neq \mathbf{0}$ also.

Students' Solutions Manual for
Chapter 7: Linearity

7.1.1. (b) Not linear; (d) linear; (f) not linear.

7.1.2. (a) $F(0,0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\quad (c)$ $F(-x,-y) = F(x,y) \neq -F(x,y)$.

7.1.3. (b) Not linear; (d) linear; (f) linear.

7.1.5. (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, $\quad (d)$ $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\quad (f)$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

7.1.9. No, because linearity would require

$$L\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = L\left[\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right] = L\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - L\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 3 \neq -2.$$

7.1.15. (a) $L[\,c\,X + d\,Y\,] = A(c\,X + d\,Y) = c\,A\,X + d\,A\,Y = c\,L[\,X\,] + d\,L[\,Y\,]$;

matrix representative: $\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$.

7.1.16. (b) Not linear; codomain space $= \mathcal{M}_{n\times n}$. $\quad (d)$ Not linear; codomain space $= \mathcal{M}_{n\times n}$.
(f) Linear; codomain space $= \mathbb{R}$.

7.1.19. (b) Not linear; codomain $= \mathbb{R}$. $\quad (d)$ Linear; codomain $= \mathbb{R}$.
(f) Linear; codomain $= \mathrm{C}^1(\mathbb{R})$. $\quad (h)$ Linear; codomain $= \mathrm{C}^0(\mathbb{R})$.

7.1.22.
$$\begin{aligned} I_w[\,c\,f + d\,g\,] &= \int_a^b \big[\,c\,f(x) + d\,g(x)\,\big]\,w(x)\,dx \\ &= c\int_a^b f(x)\,w(x)\,dx + d\int_a^b g(x)\,w(x)\,dx = c\,I_w[\,f\,] + d\,I_w[\,g\,]. \end{aligned}$$

7.1.24.
$$\begin{aligned} \Delta[\,c\,f + d\,g\,] &= \frac{\partial^2}{\partial x^2}\big[\,c\,f(x,y) + d\,g(x,y)\,\big] + \frac{\partial^2}{\partial y^2}\big[\,c\,f(x,y) + d\,g(x,y)\,\big] \\ &= c\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + d\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right) = c\,\Delta[\,f\,] + d\,\Delta[\,g\,]. \end{aligned}$$

7.1.26. (a) Gradient: $\nabla(c\,f + d\,g) = c\,\nabla f + d\,\nabla g$; domain is space of continuously differentiable scalar functions; codomain is space of continuous vector fields.

7.1.27. (b) dimension $= 4$; basis: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(d) dimension $= 4$; basis given by L_0, L_1, L_2, L_3, where $L_i[\,a_3\,x^3 + a_2\,x^2 + a_1\,x + a_0\,] = a_i$.

7.1.28. True. The dimension is 2, with basis $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

7.1.30. (a) $\mathbf{a} = (\,3, -1, 2\,)^T$, $\quad (c)$ $\mathbf{a} = \left(\,\frac{5}{4}, -\frac{1}{2}, \frac{5}{4}\,\right)^T$.

7.1.34. (a) $9 - 36\,x + 30\,x^2$, $\quad (c)$ 1.

7.1.37. (a) $S \circ T = T \circ S =$ clockwise rotation by 60° $=$ counterclockwise rotation by 300°;
(b) $S \circ T = T \circ S =$ reflection in the line $y = x$; (c) $S \circ T = T \circ S =$ rotation by 180°;
(e) $S \circ T = T \circ S = \mathrm{O}$.

7.1.39. (a) $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

(b) $R \circ S = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \neq S \circ R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Under $R \circ S$, the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ go to $\mathbf{e}_3, -\mathbf{e}_1, -\mathbf{e}_2$, respectively.
Under $S \circ R$, they go to $\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1$.

7.1.41. (a) $L = E \circ D$ where $D[\,f(x)\,] = f'(x)$, $E[\,g(x)\,] = g(0)$. No, they do not commute: $D \circ E$ is not even defined since the codomain of E, namely $\mathbb{R}$, is not the domain of D, the space of differentiable functions. (b) $e = 0$ is the only condition.

7.1.43. Given $L = a_n\,D^n + \cdots + a_1\,D + a_0$, $M = b_n\,D^n + \cdots + b_1\,D + b_0$, with a_i, b_i constant, the linear combination $c\,L + d\,M = (c\,a_n + d\,b_n)\,D^n + \cdots + (c\,a_1 + d\,b_1)\,D + (c\,a_0 + d\,b_0)$, is also a constant coefficient linear differential operator, proving that it is a subspace of the space of all linear operators. A basis is $D^n, D^{n-1}, \ldots, D, 1$ and so its dimension is $n + 1$.

7.1.46. If $p(x, y) = \sum c_{ij}\,x^i y^j$ then $p(x, y) = \sum c_{ij}\,\partial_x^i \partial_y^j$ is a linear combination of linear operators, which can be built up as compositions $\partial_x^i \circ \partial_y^j = \partial_x \circ \cdots \circ \partial_x \circ \partial_y \circ \cdots \circ \partial_y$ of the basic first order linear partial differential operators.

7.1.51. (a) The inverse is the scaling transformation that halves the length of each vector.
(b) The inverse is counterclockwise rotation by 45°. $\quad (d)$ No inverse.

7.1.52. (b) Function: $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$; inverse: $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

(d) Function: $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$; no inverse.

♢ 7.1.55. If $L \circ M = L \circ N = \mathrm{I}_W$, $M \circ L = N \circ L = \mathrm{I}_V$, then, by associativity,
$$M = M \circ \mathrm{I}_W = M \circ (L \circ N) = (M \circ L) \circ N = \mathrm{I}_V \circ N = N.$$

♡ 7.1.58. (a) Every vector in V can be uniquely written as a linear combination of the basis elements: $\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n$. Assuming linearity, we compute
$$L[\mathbf{v}] = L[c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n] = c_1 L[\mathbf{v}_1] + \cdots + c_n L[\mathbf{v}_n] = c_1 \mathbf{w}_1 + \cdots + c_n \mathbf{w}_n.$$
Since the coefficients $c_1, \ldots, c_n$ of $\mathbf{v}$ are uniquely determined, this formula serves to uniquely define the function $L : V \to W$. We must then check that the resulting function is linear. Given any two vectors $\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n$, $\mathbf{w} = d_1 \mathbf{v}_1 + \cdots + d_n \mathbf{v}_n$ in V, we have
$$L[\mathbf{v}] = c_1 \mathbf{w}_1 + \cdots + c_n \mathbf{w}_n, \qquad L[\mathbf{w}] = d_1 \mathbf{w}_1 + \cdots + d_n \mathbf{w}_n.$$
Then, for any $a, b \in \mathbb{R}$,
$$\begin{aligned} L[a\mathbf{v} + b\mathbf{w}] &= L[(a c_1 + b d_1)\mathbf{v}_1 + \cdots + (a c_n + b d_n)\mathbf{v}_n] \\ &= (a c_1 + b d_1)\mathbf{w}_1 + \cdots + (a c_n + b d_n)\mathbf{w}_n \\ &= a\big(c_1 \mathbf{w}_1 + \cdots + c_n \mathbf{w}_n\big) + b\big(d_1 \mathbf{w}_1 + \cdots + d_n \mathbf{w}_n\big) = a L[\mathbf{v}] + d L[\mathbf{w}], \end{aligned}$$
proving linearity of L.

(b) The inverse is uniquely defined by the requirement that $L^{-1}[\mathbf{w}_i] = \mathbf{v}_i$, $i = 1, \ldots, n$. Note that $L \circ L^{-1}[\mathbf{w}_i] = L[\mathbf{v}_i] = \mathbf{w}_i$, and hence $L \circ L^{-1} = \mathrm{I}_W$ since $\mathbf{w}_1, \ldots, \mathbf{w}_n$ is a basis. Similarly, $L^{-1} \circ L[\mathbf{v}_i] = L^{-1}[\mathbf{w}_i] = \mathbf{v}_i$, and so $L^{-1} \circ L = \mathrm{I}_V$.

(c) If $A = (\,\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n\,)$, $B = (\,\mathbf{w}_1\ \mathbf{w}_2\ \ldots\ \mathbf{w}_n\,)$, then L has matrix representative $B A^{-1}$, while L^{-1} has matrix representative $A B^{-1}$.

(d) (i) $L = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, $L^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$.

7.1.63. (a) $L[a x^2 + b x + c] = a x^2 + (b + 2a) x + (c + b)$;
$$L^{-1}[a x^2 + b x + c] = a x^2 + (b - 2a) x + (c - b + 2a) = e^{-x} \int_{-\infty}^{x} e^{y}\, p(y)\, dy.$$

7.2.1. (a) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. (i) The line $y = x$; (ii) the rotated square $0 \le x + y, x - y \le \sqrt{2}$; (iii) the unit disk.

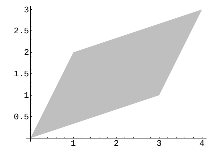
(c) $\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$. (i) The line $4x + 3y = 0$; (ii) the rotated square with vertices $(0,0)^T, \left(\frac{4}{5}, \frac{3}{5}\right)^T, \left(\frac{1}{5}, \frac{7}{5}\right)^T, \left(-\frac{3}{5}, \frac{4}{5}\right)^T$; (iii) the unit disk.

7.2.2. (a) $L^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a rotation by $\theta = \pi$;

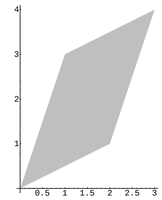
(b) L is clockwise rotation by 90°, or, equivalently, counterclockwise rotation by 270°.

7.2.5. The image is the line that goes through the image points $\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ -1 \end{pmatrix}$.

7.2.6. Parallelogram with vertices $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$:

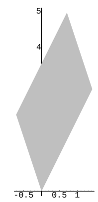


(b) Parallelogram with vertices $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$:

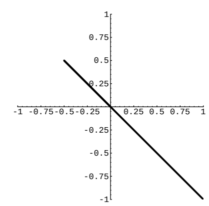


(d) Parallelogram with vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} - \sqrt{2} \\ \frac{1}{\sqrt{2}} + \sqrt{2} \end{pmatrix}, \begin{pmatrix} -\frac{3}{\sqrt{2}} + 2\sqrt{2} \\ \frac{3}{\sqrt{2}} + 2\sqrt{2} \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ 2\sqrt{2} \end{pmatrix}:$$



(f) Line segment between $\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$:



7.2.9. (b) True. (d) False: in general circles are mapped to ellipses.

7.2.13. (b) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

A shear of magnitude -1 along the x-axis, followed by a scaling in the y direction by a factor of 2, followed by a shear of magnitude -1 along the y-axis.

(d) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$:

A shear of magnitude 1 along the x-axis that fixes the xz-plane, followed a shear of magnitude -1 along the y-axis that fixes the xy plane, followed by a reflection in the xz plane, followed by a scaling in the z direction by a factor of 2, followed a shear of magnitude -1 along the z-axis that fixes the xz-plane, followed a shear of magnitude 1 along the y-axis that fixes the yz plane.

7.2.15. (b) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

7.2.17. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the identity transformation; $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is a reflection in the plane $x = z$; $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is rotation by 120° around the line $x = y = z$.

7.2.19. $\det \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = +1$, representing a 180° rotation, while $\det \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1$,
and so is a reflection — but through the origin, not a plane, since it doesn't fix any nonzero vectors.

♢ 7.2.21. (a) First, $\mathbf{w} = (\mathbf{u}\cdot\mathbf{v})\,\mathbf{u} = (\mathbf{u}^T\mathbf{v})\,\mathbf{u} = \mathbf{u}\,\mathbf{u}^T\mathbf{v}$ is the orthogonal projection of $\mathbf{v}$ onto the line in the direction of $\mathbf{u}$. So the reflected vector is $\mathbf{v} - 2\,\mathbf{w} = (\mathrm{I} - 2\,\mathbf{u}\,\mathbf{u}^T)\mathbf{v}$.

(c) (i) $\begin{pmatrix} \frac{7}{25} & 0 & -\frac{24}{25} \\ 0 & 1 & 0 \\ -\frac{24}{25} & 0 & -\frac{7}{25} \end{pmatrix}$.

7.2.24. (b) $\begin{pmatrix} 1 & -6 \\ -\frac{4}{3} & 3 \end{pmatrix}$, (d) $\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$.

7.2.25. (a) $\begin{pmatrix} -3 & -1 & -2 \\ 6 & 1 & 6 \\ 1 & 1 & 0 \end{pmatrix}$.

7.2.26. (b) bases: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; canonical form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$;

(d) bases: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$; canonical form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

♢ 7.2.28. (a) Let Q have columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$, so Q is an orthogonal matrix. Then the matrix representative in the orthonormal basis is

$$B = Q^{-1} A Q = Q^T A Q, \quad \text{and} \quad B^T = Q^T A^T (Q^T)^T = Q^T A Q = B.$$

(b) Not necessarily. For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $S = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, then $S^{-1} A S = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ is not symmetric.

7.3.1. (b) True. (d) False: in general circles are mapped to ellipses.

7.3.3. (a) (i) The horizontal line $y = -1$; (ii) the disk $(x-2)^2 + (y+1)^2 \le 1$ of radius 1 centered at $(2, -1)^T$; (iii) the square $\{2 \le x \le 3, -1 \le y \le 0\}$.
(c) (i) The horizontal line $y = 2$; (ii) the elliptical domain $x^2 - 4xy + 5y^2 + 6x - 16y + 12 \le 0$; (iii) the parallelogram with vertices $(1,2)^T, (2,2)^T, (4,3)^T, (3,3)^T$.

7.3.4. (a) $T_3 \circ T_4[\mathbf{x}] = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$,

with $\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$;

(c) $T_3 \circ T_6[\mathbf{x}] = \begin{pmatrix} \frac32 & \frac32 \\ \frac12 & \frac12 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 2 \end{pmatrix},$

with $\begin{pmatrix} \frac32 & \frac32 \\ \frac12 & \frac12 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

7.3.7. (a) $F[\mathbf{x}] = A\,\mathbf{x} + \mathbf{b}$ has an inverse if and only if A in nonsingular.
(b) Yes: $F^{-1}[\mathbf{x}] = A^{-1}\mathbf{x} - A^{-1}\mathbf{b}$.
(c) $T_3^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad T_4^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix},$
T_6 has no inverse.

7.3.11. (a) Isometry, (c) not an isometry.

7.3.15. (b) $F[\mathbf{x}] = \begin{pmatrix} \frac{\sqrt3}{2} & -\frac12 \\ \frac12 & \frac{\sqrt3}{2} \end{pmatrix} \left[\mathbf{x} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt3}{2} & -\frac12 \\ \frac12 & \frac{\sqrt3}{2} \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{3-\sqrt3}{2} \\ \frac{1-\sqrt3}{2} \end{pmatrix};$

$G[\mathbf{x}] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\mathbf{x} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right] + \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ 3 \end{pmatrix};$

$F \circ G[\mathbf{x}] = \begin{pmatrix} -\frac12 & -\frac{\sqrt3}{2} \\ \frac{\sqrt3}{2} & -\frac12 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\sqrt3 \\ \sqrt3 \end{pmatrix} = \begin{pmatrix} -\frac12 & -\frac{\sqrt3}{2} \\ \frac{\sqrt3}{2} & -\frac12 \end{pmatrix} \left[\mathbf{x} - \begin{pmatrix} \frac{-1-\sqrt3}{2} \\ \frac{-1+\sqrt3}{2} \end{pmatrix} \right] + \begin{pmatrix} \frac{-1-\sqrt3}{2} \\ \frac{-1+\sqrt3}{2} \end{pmatrix}$

is counterclockwise rotation around the point $\begin{pmatrix} \frac{-1-\sqrt3}{2} \\ \frac{-1+\sqrt3}{2} \end{pmatrix}$ by 120°;

$G \circ F[\mathbf{x}] = \begin{pmatrix} -\frac12 & -\frac{\sqrt3}{2} \\ \frac{\sqrt3}{2} & -\frac12 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{-3+\sqrt3}{2} \\ \frac{9-\sqrt3}{2} \end{pmatrix} = \begin{pmatrix} -\frac12 & -\frac{\sqrt3}{2} \\ \frac{\sqrt3}{2} & -\frac12 \end{pmatrix} \left[\mathbf{x} - \begin{pmatrix} \frac{-1-\sqrt3}{2} \\ \frac{5-\sqrt3}{2} \end{pmatrix} \right] + \begin{pmatrix} \frac{-1-\sqrt3}{2} \\ \frac{5-\sqrt3}{2} \end{pmatrix}$

is counterclockwise rotation around the point $\begin{pmatrix} \frac{-1-\sqrt3}{2} \\ \frac{5-\sqrt3}{2} \end{pmatrix}$ by 120°.

♡ 7.3.16. (a) If $F[\mathbf{x}] = Q\,\mathbf{x} + \mathbf{a}$ and $G[\mathbf{x}] = R\,\mathbf{x} + \mathbf{b}$, then $G \circ F[\mathbf{x}] = R\,Q\,\mathbf{x} + (R\,\mathbf{a} + \mathbf{b}) = S\,\mathbf{x} + \mathbf{c}$ is a isometry since $S = Q\,R$, the product of two orthogonal matrices, is also an orthogonal matrix. (b) $F[\mathbf{x}] = \mathbf{x} + \mathbf{a}$ and $G[\mathbf{x}] = \mathbf{x} + \mathbf{b}$, then $G \circ F[\mathbf{x}] = \mathbf{x} + (\mathbf{a} + \mathbf{b}) = \mathbf{x} + \mathbf{c}$.

♢ 7.3.17. (a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} x + 2 \\ -y \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{3}{\sqrt2} \\ \frac{3}{\sqrt2} \end{pmatrix} = \begin{pmatrix} y + \frac{3}{\sqrt2} \\ x + \frac{3}{\sqrt2} \end{pmatrix}$.

♢ 7.3.21. First, if L is an isometry, and $\|\,\mathbf{u}\,\| = 1$ then $\|\,L[\mathbf{u}]\,\| = 1$, proving that $L[\mathbf{u}] \in S_1$. Conversely, if L preserves the unit sphere, and $\mathbf{0} \neq \mathbf{v} \in V$, then $\mathbf{u} = \mathbf{v}/r \in S_1$ where $r = \|\,\mathbf{v}\,\|$, so $\|\,L[\mathbf{v}]\,\| = \|\,L[r\,\mathbf{v}]\,\| = \|\,r\,L[\mathbf{u}]\,\| = r\,\|\,L[\mathbf{u}]\,\| = r = \|\,\mathbf{v}\,\|$, proving (7.40).

♡ 7.3.24. (a) $q(H\,\mathbf{x}) = (x\cosh\alpha + y\sinh\alpha)^2 - (x\sinh\alpha + y\cosh\alpha)^2$
$= (\cosh^2\alpha - \sinh^2\alpha)(x^2 - y^2) = x^2 - y^2 = q(\mathbf{x}).$

7.4.2. (a) $L(x) = 3x$; domain $\mathbb{R}$; codomain $\mathbb{R}$; right-hand side -5; inhomogeneous.

(c) $L(u,v,w) = \begin{pmatrix} u-2v \\ v-w \end{pmatrix}$; domain $\mathbb{R}^3$; codomain $\mathbb{R}^2$; right-hand side $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$; inhomogeneous.

(e) $L[u] = u'(x) + 3x\,u(x)$; domain $C^1(\mathbb{R})$; codomain $C^0(\mathbb{R})$; right-hand side 0; homogeneous.

(g) $L[u] = \begin{pmatrix} u'(x) - u(x) \\ u(0) \end{pmatrix}$; domain $C^1(\mathbb{R})$; codomain $C^0(\mathbb{R}) \times \mathbb{R}$; right-hand side $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$; inhomogeneous.

(k) $L[u,v] = \begin{pmatrix} u''(x) - v''(x) - 2u(x) + v(x) \\ u(0) - v(0) \\ u(1) - v(1) \end{pmatrix}$; domain $C^2(\mathbb{R}) \times C^2(\mathbb{R})$; codomain $C^0(\mathbb{R}) \times \mathbb{R}^2$; right-hand side $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; homogeneous.

(m) $L[u] = \int_0^\infty u(t)\, e^{-s\,t}\, dt$; domain $C^0(\mathbb{R})$; codomain $C^0(\mathbb{R})$; right-hand side $1 + s^2$; inhomogeneous.

7.4.3. $L[u] = u(x) + \int_a^b K(x,y)\, u(y)\, dy$. The domain is $C^0(\mathbb{R})$ and the codomain is $\mathbb{R}$. To show linearity, for constants c, d,

$$\begin{aligned} L[c\,u + d\,v] &= [\,c\,u(x) + d\,v(x)\,] + \int_a^b K(x,y)\,[\,c\,u(y) + d\,v(y)\,]\, dy \\ &= c\left(u(x) + \int_a^b K(x,y)\, u(y)\, dy \right) + d\left(v(x) + \int_a^b K(x,y)\, v(y)\, dy \right) = c\,L[u] + d\,L[v]. \end{aligned}$$

7.4.6. (a) $u(x) = c_1 e^{2x} + c_2 e^{-2x}$, $\dim = 2$; (c) $u(x) = c_1 + c_2 e^{3x} + c_3 e^{-3x}$, $\dim = 3$.

7.4.9. (a) $p(D) = D^3 + 5D^2 + 3D - 9$.
(b) $e^x, e^{-3x}, x\,e^{-3x}$. The general solution is $y(x) = c_1 e^x + c_2 e^{-3x} + c_2\, x\, e^{-3x}$.

7.4.10. (a) Minimal order 2: $u'' + u' - 6u = 0$. (c) minimal order 2: $u'' - 2u' + u = 0$.

7.4.11. (a) $u = c_1 x + \dfrac{c_2}{x^5}$, (c) $u = c_1 |\,x\,|^{(1+\sqrt{5})/2} + c_2 |\,x\,|^{(1-\sqrt{5})/2}$.

♢ 7.4.14. (b) (i) $u(x) = c_1 x + c_2 x \log |\,x\,|$

7.4.15. $v'' - 4v = 0$, so $u(x) = c_1 \dfrac{e^{2x}}{x} + c_2 \dfrac{e^{-2x}}{x}$. The solutions with $c_1 + c_2 = 0$ are continuously differentiable at $x = 0$, but only the zero solution is twice continuously differentiable.

7.4.18. $u = c_1 + c_2 \log r$. The solutions form a two-dimensional vector space.

7.4.21. (a) Basis: $1,\ x,\ y,\ z,\ x^2 - y^2,\ x^2 - z^2,\ x\,y,\ x\,z,\ y\,z$; dimension $= 9$.

7.4.24. (a) Not in the image. (b) $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -\frac{7}{5} \\ -\frac{6}{5} \\ 1 \end{pmatrix}$.

7.4.25. (b) $x = -\frac{1}{7} + \frac{3}{7}z,\ y = \frac{4}{7} + \frac{2}{7}z$, not unique; ($d$) $u = 2, v = -1, w = 0$, unique.

7.4.26. (b) $u(x) = \frac{1}{6}e^x \sin x + c_1 e^{2x/5}\cos\frac{4}{5}x + c_2 e^{2x/5}\sin\frac{4}{5}x$.

7.4.27. (b) $u(x) = \frac{1}{4} - \frac{1}{4}\cos 2x$, ($d$) $u(x) = -\frac{1}{10}\cos x + \frac{1}{5}\sin x + \frac{11}{10}e^{-x}\cos 2x + \frac{9}{10}e^{-x}\sin 2x$.

7.4.28. (b) $u(x) = \frac{1}{2}\log x + \frac{3}{4} + c_1 x + c_2 x^2$.

7.4.29. (a) Unique solution: $u(x) = x - \pi\,\dfrac{\sin\sqrt{2}\,x}{\sin\sqrt{2}\,\pi}$;

(d) infinitely many solutions: $u(x) = \frac{1}{2} + c\,e^{-x}\sin x$; ($f$) no solution.

7.4.32. (b) $u(x) = -\frac{1}{9}x - \frac{1}{10}\sin x + c_1 e^{3x} + c_2 e^{-3x}$,

(d) $u(x) = \frac{1}{6}x\,e^x - \frac{1}{18}e^x + \frac{1}{4}e^{-x} + c_1 e^x + c_2 e^{-2x}$.

7.4.35. $u(x) = -7\cos\sqrt{x} - 3\sin\sqrt{x}$.

7.4.36. (a) $u(x) = \frac{1}{9}x + \cos 3x + \frac{1}{27}\sin 3x$, ($c$) $u(x) = 3\cos 2x + \frac{3}{10}\sin 2x - \frac{1}{5}\sin 3x$.

7.4.41. (b) $u(x) = c_1 e^{-3x}\cos x + c_2 e^{-3x}\sin x$,

(d) $u(x) = c_1 e^{x/\sqrt{2}}\cos\frac{1}{\sqrt{2}}x + c_2 e^{-x/\sqrt{2}}\cos\frac{1}{\sqrt{2}}x + c_3 e^{x/\sqrt{2}}\sin\frac{1}{\sqrt{2}}x + c_4 e^{-x/\sqrt{2}}\sin\frac{1}{\sqrt{2}}x$.

7.4.42. (a) Minimal order 2: $u'' + 2u' + 10u = 0$;

(c) minimal order 5: $u^{(v)} + 4u^{(iv)} + 14u''' + 20u'' + 25u' = 0$.

7.4.43. (b) $u(x) = c_1 e^x + c_2 e^{(\mathrm{i}-1)x} = \left(c_1 e^x + c_2 e^{-x}\cos x\right) + \mathrm{i}\,e^{-x}\sin x$.

♡ 7.4.46. (a) $\dfrac{\partial u}{\partial t} = -k^2 e^{-k^2 t + \mathrm{i}kx} = \dfrac{\partial^2 u}{\partial x^2}$; ($c$) $e^{-k^2 t}\cos kx,\ e^{-k^2 t}\sin kx$.

7.4.48. (a) Conjugated, (d) not conjugated.

♢ 7.4.51. $L[\mathbf{u}] = L[\mathbf{v}] + \mathrm{i}\,L[\mathbf{w}] = \mathbf{f}$, and, since L is real, the real and imaginary parts of this equation yield $L[\mathbf{v}] = \mathbf{f}$, $L[\mathbf{w}] = \mathbf{0}$.

7.5.1. (a) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, ($c$) $\begin{pmatrix} \frac{13}{7} & -\frac{10}{7} \\ \frac{5}{7} & \frac{15}{7} \end{pmatrix}$.

7.5.2. Domain (a), codomain (b): $\begin{pmatrix} 2 & -3 \\ 4 & 9 \end{pmatrix}$; domain (b), codomain (c): $\begin{pmatrix} \frac{3}{2} & -\frac{5}{2} \\ \frac{1}{3} & \frac{10}{3} \end{pmatrix}$;

domain (c), codomain (a): $\begin{pmatrix} \frac{6}{7} & -\frac{1}{7} \\ \frac{5}{7} & \frac{5}{7} \end{pmatrix}$.

7.5.3. (b) $\begin{pmatrix} 1 & -2 & 0 \\ \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & \frac{2}{3} & 2 \end{pmatrix}$

7.5.4. Domain (a), codomain (b): $\begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & -3 \\ 0 & 2 & 6 \end{pmatrix}$; domain (b), codomain (c): $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ \frac{1}{3} & \frac{4}{3} & \frac{5}{3} \end{pmatrix}$.

7.5.5. Domain (a), codomain (a): $\begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix}$; domain (a), codomain (c): $\begin{pmatrix} 2 & 0 & -2 \\ 8 & 8 & 4 \end{pmatrix}$.

♢ 7.5.8. (a) Given $\mathbf{u} \in U$, $\mathbf{v} \in V$, we have $\langle\, \mathbf{u}\,, (L+M)^*[\mathbf{v}\,]\,\rangle = \langle\!\langle\, (L+M)[\mathbf{u}\,]\,, \mathbf{v}\,\rangle\!\rangle =$ $\langle\!\langle\, L[\mathbf{u}\,]\,, \mathbf{v}\,\rangle\!\rangle + \langle\!\langle\, M[\mathbf{u}\,]\,, \mathbf{v}\,\rangle\!\rangle = \langle\, \mathbf{u}\,, L^*[\mathbf{v}\,]\,\rangle + \langle\, \mathbf{u}\,, M^*[\mathbf{v}\,]\,\rangle = \langle\, \mathbf{u}\,, (L^*+M^*)[\mathbf{v}\,]\,\rangle$.
Since this holds for all $\mathbf{u} \in U$, $\mathbf{v} \in V$, we conclude that $(L+M)^* = L^* + M^*$.
(b) $\langle\, \mathbf{u}\,, (cL)^*[\mathbf{v}\,]\,\rangle = \langle\!\langle\, (cL)[\mathbf{u}\,]\,, \mathbf{v}\,\rangle\!\rangle = c\,\langle\!\langle\, L[\mathbf{u}\,]\,, \mathbf{v}\,\rangle\!\rangle = c\,\langle\, \mathbf{u}\,, L^*[\mathbf{v}\,]\,\rangle = \langle\, \mathbf{u}\,, c\,L^*[\mathbf{v}\,]\,\rangle$.

7.5.11. In all cases, $L = L^*$ if and only if its matrix representative A, with respect to the standard basis, is symmetric. (a) $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = A^T$, (c) $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = A^T$.

7.5.14. (a) $a_{12} = \frac{1}{2}\,a_{21}$, $a_{13} = \frac{1}{3}\,a_{31}$, $\frac{1}{2}\,a_{23} = \frac{1}{3}\,a_{32}$, (b) Example: $\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{pmatrix}$.

7.5.18. (a) $(L + L^*)^* = L^* + (L^*)^* = L^* + L$. (b) Since $L \circ L^* = (L^*)^* \circ L^*$, this follows from Theorem 7.60. (Or it can be proved directly.)

7.5.21. (a) $$\langle\, M_a[u]\,, v\,\rangle = \int_a^b M_a[u(x)]\,v(x)\,dx = \int_a^b a(x)\,u(x)\,v(x)\,dx = \int_a^b u(x)\,M_a[v(x)]\,dx$$ $$= \langle\, u\,, M_a[v]\,\rangle, \text{ proving self-adjointness.}$$

7.5.24. Minimizer: $\left(\frac{1}{5}, -\frac{1}{5}\right)^T$; minimum value: $-\frac{1}{5}$.

7.5.26. Minimizer: $\left(\frac{2}{3}, \frac{1}{3}\right)^T$; minimum value: -2.

7.5.28. (a) Minimizer: $\left(\frac{7}{13}, \frac{2}{13}\right)^T$; minimum value: $-\frac{7}{26}$.
(c) Minimizer: $\left(\frac{12}{13}, \frac{5}{26}\right)^T$; minimum value: $-\frac{43}{52}$.

7.5.29. (a) $\frac{1}{3}$, (b) $\frac{6}{11}$.

Students' Solutions Manual for

Chapter 8: Eigenvalues and Singular Values

8.1.1. (a) $u(t) = -3\,e^{5\,t}$, (b) $u(t) = 3\,e^{2(t-1)}$.

8.1.2. $\gamma = \log 2/100 \approx .0069$. After 10 years: 93.3033 gram; after 100 years: 50 gram; after 1000 years: .0977 gram.

8.1.5. The solution is $u(t) = u(0)\,e^{1.3\,t}$. To double, we need $e^{1.3\,t} = 2$, so $t = \log 2/1.3 = .5332$.
To quadruple takes twice as long, $t = 1.0664$.
To reach 2 million, the colony needs $t = \log 10^6/1.3 = 10.6273$.

◇ 8.1.7. (a) If $u(t) \equiv u_\star = -\dfrac{b}{a}$, then $\dfrac{du}{dt} = 0 = a\,u + b$, hence it is a solution.

(b) $v = u - u_\star$ satisfies $\dfrac{dv}{dt} = a\,v$, so $v(t) = c\,e^{a\,t}$, and $u(t) = c\,e^{a\,t} - \dfrac{b}{a}$.

8.1.8. (a) $u(t) = \frac{1}{2} + \frac{1}{2}\,e^{2\,t}$.

8.1.11. (a) $u(t) = \frac{1}{3}\,e^{2\,t/7}$. (b) One unit: $t = \log\big[\,1/(1/3 - .3333)\,\big]/(2/7) = 36.0813$;
1000 units: $t = \log\big[\,1000/(1/3 - .3333)\,\big]/(2/7) = 60.2585$;

(c) One unit: $t \approx 30.2328$ solves $\frac{1}{3}\,e^{2\,t/7} - .3333\,e^{.2857\,t} = 1$.
1000 units: $t \approx 52.7548$ solves $\frac{1}{3}\,e^{2\,t/7} - .3333\,e^{.2857\,t} = 1000$.

Note: The solutions to these nonlinear equations are found by a numerical equation solver, e.g., the bisection method, or Newton's method, [**8**].

8.2.1. (a) Eigenvalues: $3, -1$; eigenvectors: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(e) Eigenvalues: $4, 3, 1$; eigenvectors: $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

(g) Eigenvalues: $0, 1 + \mathrm{i}\,, 1 - \mathrm{i}\,$; eigenvectors: $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 - 2\,\mathrm{i} \\ 3 - \mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} 3 + 2\,\mathrm{i} \\ 3 + \mathrm{i} \\ 1 \end{pmatrix}$.

(i) -1 is a simple eigenvalue, with eigenvector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$;

2 is a double eigenvalue, with eigenvectors $\begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix}$.

8.2.4. (a) O is a trivial example.

8.2.7. (*a*) Eigenvalues: $\mathrm{i}\,,\ -1+\mathrm{i}\,;$ eigenvectors: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(*c*) Eigenvalues: $-3,\ 2\,\mathrm{i}\,;$ eigenvectors: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{3}{5}+\frac{1}{5}\,\mathrm{i} \\ 1 \end{pmatrix}$.

8.2.9. For $n=2$, the eigenvalues are $0,2$, and the eigenvectors are $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $n=3$, the eigenvalues are $0,0,3$, and the eigenvectors are $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

♢ 8.2.10. (*a*) If $A\mathbf{v}=\lambda\mathbf{v}$, then $A(c\mathbf{v})=c\,A\mathbf{v}=c\lambda\mathbf{v}=\lambda(c\mathbf{v})$ and so $c\mathbf{v}$ satisfies the eigenvector equation for the eigenvalue λ. Moreover, since $\mathbf{v}\neq\mathbf{0}$, also $c\mathbf{v}\neq\mathbf{0}$ for $c\neq 0$, and so $c\mathbf{v}$ is a bona fide eigenvector. (*b*) If $A\mathbf{v}=\lambda\mathbf{v}$, $A\mathbf{w}=\lambda\mathbf{w}$, then

$$A(c\mathbf{v}+d\mathbf{w})=c\,A\mathbf{v}+d\,A\mathbf{w}=c\lambda\mathbf{v}+d\lambda\mathbf{w}=\lambda(c\mathbf{v}+d\mathbf{w}).$$

8.2.12. True — by the same computation as in Exercise 8.2.10(*a*), $c\mathbf{v}$ is an eigenvector for the same (real) eigenvalue λ.

8.2.15. (*a*) $\operatorname{tr} A=2=3+(-1);\ \det A=-3=3\cdot(-1)$.

(*e*) $\operatorname{tr} A=8=4+3+1;\ \det A=12=4\cdot 3\cdot 1$.

(*g*) $\operatorname{tr} A=2=0+(1+\mathrm{i}\,)+(1-\mathrm{i}\,);\ \det A=0=0\cdot(1+\mathrm{i}\sqrt{2})\cdot(1-\mathrm{i}\sqrt{2})$.

(*i*) $\operatorname{tr} A=3=(-1)+2+2;\ \det A=-4=(-1)\cdot 2\cdot 2$.

8.2.17. If U is upper triangular, so is $U-\lambda\,\mathrm{I}$, and hence $p(\lambda)=\det(U-\lambda\,\mathrm{I}\,)$ is the product of the diagonal entries, so $p(\lambda)=\prod_i (u_{ii}-\lambda)$. Thus, the roots of the characteristic equation are $u_{11},\ldots,u_{nn}$ — the diagonal entries of U.

8.2.21. (*a*) False. For example, 0 is an eigenvalue of both $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, but the eigenvalues of $A+B=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are $\pm\,\mathrm{i}\,$.

♢ 8.2.24. (*a*) Starting with $A\mathbf{v}=\lambda\mathbf{v}$, multiply both sides by A^{-1} and divide by λ to obtain $A^{-1}\mathbf{v}=(1/\lambda)\,\mathbf{v}$. Therefore, $\mathbf{v}$ is an eigenvector of A^{-1} with eigenvalue $1/\lambda$.

(*b*) If 0 is an eigenvalue, then A is not invertible.

8.2.29. (*b*) False. For example, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $\mathrm{i}\,,-\mathrm{i}\,$, whereas $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has eigenvalues $1,-1$.

8.2.31. (*a*) (*ii*) $Q=\begin{pmatrix} \frac{7}{25} & -\frac{24}{25} \\ -\frac{24}{25} & -\frac{7}{25} \end{pmatrix}$. Eigenvalues $-1,1$; eigenvectors $\begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}, \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{pmatrix}$.

♢ 8.2.32. (a) $\det(B - \lambda\,\mathrm{I}) = \det(S^{-1} A S - \lambda\,\mathrm{I}) = \det\big[\, S^{-1}(A - \lambda\,\mathrm{I}) S\,\big]$
$$= \det S^{-1} \det(A - \lambda\,\mathrm{I}) \det S = \det(A - \lambda\,\mathrm{I}).$$
(b) The eigenvalues are the roots of the common characteristic equation. (c) Not usually. If $\mathbf{w}$ is an eigenvector of B, then $\mathbf{v} = S\,\mathbf{w}$ is an eigenvector of A and conversely.

8.2.36. (a) The characteristic equation of a 3×3 matrix is a real cubic polynomial, and hence has at least one real root. (b) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ has eigenvalues $\pm\,\mathrm{i}$.

8.2.39. False. For example, $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ has eigenvalues $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,\mathrm{i}$.

8.2.40. (a) According to Exercise 1.2.29, if $\mathbf{z} = (1, 1, \ldots, 1)^T$, then $A\,\mathbf{z}$ is the vector of row sums of A, and hence, by the assumption, $A\,\mathbf{z} = \mathbf{z}$, which means that $\mathbf{z}$ is an eigenvector with eigenvalue 1.

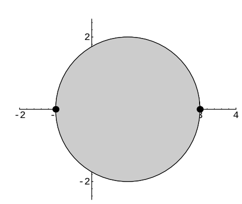
♢ 8.2.44. (a) The axis of the rotation is the eigenvector $\mathbf{v}$ corresponding to the eigenvalue $+1$. Since $Q\,\mathbf{v} = \mathbf{v}$, the rotation fixes the axis, and hence must rotate around it. Choose an orthonormal basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, where $\mathbf{u}_1$ is a unit eigenvector in the direction of the axis of rotation, while $\mathbf{u}_2 + \mathrm{i}\,\mathbf{u}_3$ is a complex eigenvector for the eigenvalue $e^{\mathrm{i}\,\theta}$. In this basis, Q has matrix form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$, where θ is the angle of rotation.

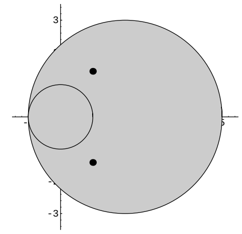
(b) The axis is the eigenvector $\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ for the eigenvalue 1. The complex eigenvalue is $\frac{7}{13} + \mathrm{i}\,\frac{2\sqrt{30}}{13}$, and so the angle is $\theta = \cos^{-1}\frac{7}{13} \approx 1.00219$.

♡ 8.2.47. (a) $M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: eigenvalues $1, -1$; eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

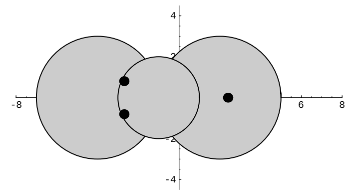
♡ 8.2.52. (a) Follows by direct computation:
$$p_A(A) = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} - (a + d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
(b) By part (a), $\mathrm{O} = A^{-1} p_A(A) = A - (\operatorname{tr} A)\,\mathrm{I} + (\det A) A^{-1}$, and the formula follows upon solving for A^{-1}. (c) $\operatorname{tr} A = 4$, $\det A = 7$ and one checks $A^2 - 4A + 7\,\mathrm{I} = \mathrm{O}$.

8.2.53. (a) Gershgorin disk: $|\,z - 1\,| \le 2$; eigenvalues: $3, -1$;



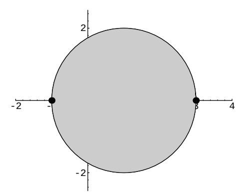


(c) Gershgorin disks: $|\, z-2 \,| \le 3$, $|\, z \,| \le 1$;
eigenvalues: $1 \pm \mathrm{i}\sqrt{2}$;

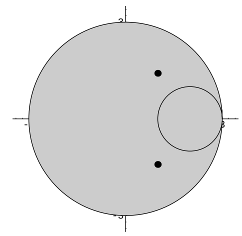


(e) Gershgorin disks: $|\, z+1 \,| \le 2$, $|\, z-2 \,| \le 3$,
$|\, z+4 \,| \le 3$; eigenvalues: $-2.69805 \pm .806289\,\mathrm{i}\,, 2.3961$;

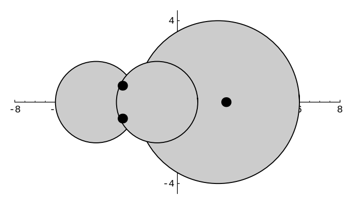
♢ 8.2.55. (i) Because A and its transpose A^T have the same eigenvalues, which must therefore belong to both D_A and D_{A^T}.



(ii) (a) Gershgorin disk: $|\, z-1 \,| \le 2$; eigenvalues: $3, -1$;



(c) Gershgorin disks: $|\, z-2 \,| \le 1$, $|\, z \,| \le 3$;
eigenvalues: $1 \pm \mathrm{i}\sqrt{2}$;



(e) Gershgorin disks: $|\, z+1 \,| \le 2$, $|\, z-2 \,| \le 4$, $|\, z+4 \,| \le 2$;
eigenvalues: $-2.69805 \pm .806289, 2.3961$;

8.2.56. (a) False: $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ is a counterexample.

8.3.1. (a) Complete; $\dim = 1$ with basis $(\,1, 1\,)^T$.
(e) Complete; $\dim = 2$ with basis $(\,1, 0, 0\,)^T, (\,0, -1, 1\,)^T$. ($g$) Not an eigenvalue.

8.3.2. (*a*) Eigenvalue: 2; eigenvector: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; not complete.

(*c*) Eigenvalues: $1 \pm 2\,\mathrm{i}$; eigenvectors: $\begin{pmatrix} 1 \pm \mathrm{i} \\ 2 \end{pmatrix}$; complete.

(*e*) Eigenvalue 3 has eigenspace basis $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$; not complete.

8.3.3. (*a*) Eigenvalues: $-2, 4$; the eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ form a basis for $\mathbb{R}^2$.

(*b*) Eigenvalues: $1 - 3\,\mathrm{i},\ 1 + 3\,\mathrm{i}$; the eigenvectors $\begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix}$, are not real, so the dimension is 0.

(*e*) The eigenvalue 1 has eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; the eigenvalue -1 has eigenvector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The eigenvectors span a two-dimensional subspace of $\mathbb{R}^3$.

8.3.4. (*a*) Complex eigenvector basis; (*b*) complex eigenvector basis;
(*e*) no eigenvector basis.

8.3.6. (*a*) True. The standard basis vectors are eigenvectors.

♢ 8.3.11. As in Exercise 8.2.32, if $\mathbf{v}$ is an eigenvector of A then $S^{-1}\mathbf{v}$ is an eigenvector of B. Moreover, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis, so do $S^{-1}\mathbf{v}_1, \ldots, S^{-1}\mathbf{v}_n$; see Exercise 2.4.21 for details.

8.3.13. In all cases, $A = S\Lambda S^{-1}$. (*a*) $S = \begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$.

(*c*) $S = \begin{pmatrix} -\frac{3}{5} + \frac{1}{5}\,\mathrm{i} & -\frac{3}{5} - \frac{1}{5}\,\mathrm{i} \\ 1 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} -1 + \mathrm{i} & 0 \\ 0 & -1 - \mathrm{i} \end{pmatrix}$.

(*e*) $S = \begin{pmatrix} 0 & 21 & 1 \\ 1 & -10 & 6 \\ 0 & 7 & 3 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(*h*) $S = \begin{pmatrix} -4 & 3 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ 0 & 6 & 0 & 0 \\ 12 & 0 & 0 & 0 \end{pmatrix}$, $\Lambda = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

8.3.16. (*a*) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{i} & -\mathrm{i} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\mathrm{i}}{2} & \frac{1}{2} & 0 \\ \frac{\mathrm{i}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$.

8.3.17. (*a*) Yes: distinct real eigenvalues $-3, 2$. (*c*) No: complex eigenvalues $1, -\frac{1}{2} \pm \frac{\sqrt{5}}{2}\,\mathrm{i}$.

8.3.18. In all cases, $A = S\Lambda S^{-1}$. (*a*) $S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 1 + \mathrm{i} & 0 \\ 0 & -1 + \mathrm{i} \end{pmatrix}$.

(*c*) $S = \begin{pmatrix} 1 & -\frac{1}{2} - \frac{1}{2}\,\mathrm{i} \\ 1 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$.

8.3.19. Use the formula $A = S\Lambda S^{-1}$. For part (*e*) you can choose any other eigenvalues and eigenvectors you want to fill in S and Λ.

$$(a)\ \begin{pmatrix} \frac{5}{3} & \frac{4}{3} \\ \frac{8}{3} & \frac{1}{3} \end{pmatrix}, \qquad (e)\ \text{example: } \begin{pmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

8.3.20. (*a*) $\begin{pmatrix} 11 & -6 \\ 18 & -10 \end{pmatrix}$.

8.3.25. Let $A = S\Lambda S^{-1}$. Then $A^2 = \mathrm{I}$ if and only if $\Lambda^2 = \mathrm{I}$, and so all its eigenvalues are ± 1. Examples: $A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$, with eigenvalues $1, -1$ and eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; or, even simpler, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with eigenvalues $1, -1$ and eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

8.4.1. (*a*) $\{\mathbf{0}\}$, the x-axis, the y-axis, the z-axis, the xy-plane, the xz-plane, the yz-plane, $\mathbb{R}^3$.

8.4.2. (*a*) $\{\mathbf{0}\}$, the line $x = y$, the line $x = -y$, $\mathbb{R}^2$;

(*c*) $\{\mathbf{0}\}$, the three lines spanned by each of the eigenvectors $(1,0,1)^T, (1,0,-1)^T, (0,1,0)^T$, the three planes spanned by pairs of eigenvectors, $\mathbb{R}^3$.

8.4.3. (*a*) Real: $\{\mathbf{0}\}$, $\mathbb{R}^2$. Complex: $\{\mathbf{0}\}$, the two (complex) lines spanned by each of the complex eigenvectors $(\mathrm{i}, 1)^T, (-\mathrm{i}, 1)^T$, $\mathbb{C}^2$.

(*d*) Real: $\{\mathbf{0}\}$, the three lines spanned by each of the eigenvectors $(1,0,0)^T, (1,0,-1)^T, (0,1,-1)^T$, the three planes spanned by pairs of eigenvectors, $\mathbb{R}^3$. Complex: $\{\mathbf{0}\}$, the three (complex) lines spanned by each of the real eigenvectors, the three (complex) planes spanned by pairs of eigenvectors, $\mathbb{C}^3$.

8.4.6. (*a*) True; (*c*) false.

8.4.8. False.

8.5.1. (*b*) Eigenvalues: $7, 3$; eigenvectors: $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(*d*) Eigenvalues: $6, 1, -4$; eigenvectors: $\begin{pmatrix} \frac{4}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{4}{5\sqrt{2}} \\ -\frac{3}{5\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

8.5.2. (*a*) Eigenvalues $\frac{5}{2} \pm \frac{1}{2}\sqrt{17}$; positive definite. (*c*) Eigenvalues $0, 1, 3$; positive semi-definite.

8.5.5. (*a*) The characteristic equation $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc) = 0$ has real roots if and only if its discriminant is non-negative: $0 \le (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc$, which is the necessary and sufficient condition for real eigenvalues.

(*b*) If A is symmetric, then $b = c$ and so the discriminant is $(a-d)^2 + 4b^2 \ge 0$.

♡ 8.5.7. (a) Let $A\mathbf{v} = \lambda\mathbf{v}$. Using the Hermitian dot product,

$$\lambda \,\|\mathbf{v}\|^2 = (A\mathbf{v})\cdot\overline{\mathbf{v}} = \mathbf{v}^T A^T \overline{\mathbf{v}} = \mathbf{v}^T \overline{A}\,\overline{\mathbf{v}} = \mathbf{v}\cdot(A\mathbf{v}) = \overline{\lambda}\,\|\mathbf{v}\|^2,$$

and hence $\lambda = \overline{\lambda}$, which implies that the eigenvalue λ is real.

(c) (ii) Eigenvalues $4, -2$; eigenvectors: $\begin{pmatrix} 2-\mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} -2+\mathrm{i} \\ 5 \end{pmatrix}$.

8.5.9. (a) Generalized eigenvalues: $\frac{5}{3}, \frac{1}{2}$; generalized eigenvectors: $\begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$.

(c) Generalized eigenvalues: $7, 1$; generalized eigenvectors: $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

8.5.13. (a) $\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt5} & -\frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ -\frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt6} & -\frac{1}{\sqrt2} & \frac{1}{\sqrt3} \\ \frac{2}{\sqrt6} & 0 & -\frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt2} & \frac{1}{\sqrt3} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt6} & \frac{2}{\sqrt6} & \frac{1}{\sqrt6} \\ -\frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt3} & -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} \end{pmatrix}$.

8.5.14. (b) $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -10 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}$,

(d) $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{5\sqrt2} & -\frac{3}{5} & -\frac{4}{5\sqrt2} \\ \frac{3}{5\sqrt2} & \frac{4}{5} & \frac{3}{5\sqrt2} \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} \frac{4}{5\sqrt2} & \frac{3}{5\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5\sqrt2} & -\frac{3}{5\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}$.

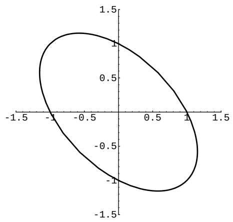
8.5.15. (a) $\begin{pmatrix} \frac{57}{25} & -\frac{24}{25} \\ -\frac{24}{25} & \frac{43}{25} \end{pmatrix}$. (c) None, since eigenvectors are not orthogonal.

8.5.17. (b) $7\left(\frac{1}{\sqrt5}x + \frac{2}{\sqrt5}y\right)^2 + 2\left(-\frac{2}{\sqrt5}x + \frac{1}{\sqrt5}y\right)^2 = \frac{7}{5}(x+2y)^2 + \frac{2}{5}(-2x+y)^2$,

(d) $\frac{1}{2}\left(\frac{1}{\sqrt3}x + \frac{1}{\sqrt3}y + \frac{1}{\sqrt3}z\right)^2 + \left(-\frac{1}{\sqrt2}y + \frac{1}{\sqrt2}z\right)^2 + 2\left(-\frac{2}{\sqrt6}x + \frac{1}{\sqrt6}y + \frac{1}{\sqrt6}z\right)^2$
$= \frac{1}{6}(x+y+z)^2 + \frac{1}{2}(-y+z)^2 + \frac{1}{3}(-2x+y+z)^2$.

8.5.21. Principal stretches = eigenvalues: $4+\sqrt3, 4-\sqrt3, 1$;
principal directions = eigenvectors: $\left(1, -1+\sqrt3, 1\right)^T, \left(1, -1-\sqrt3, 1\right)^T, (-1, 0, 1)^T$.

♢ 8.5.23. (a) Let $K = Q\Lambda Q^T$ be its spectral factorization. Then $\mathbf{x}^T K\mathbf{x} = \mathbf{y}^T\Lambda\mathbf{y}$ where $\mathbf{x} = Q\mathbf{y}$. The ellipse $\mathbf{y}^T\Lambda\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_1^2 = 1$ has its principal axes aligned with the coordinate axes and semi-axes $1/\sqrt{\lambda_i}$, $i = 1, 2$. The map $\mathbf{x} = Q\mathbf{y}$ serves to rotate the coordinate axes to align with the columns of Q, i.e., the eigenvectors, while leaving the semi-axes unchanged.

(*b*) (*ii*) 

ellipse: semi-axes $\sqrt{2}$, $\sqrt{\frac{2}{3}}$, principal axes $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

8.5.26. Only the identity matrix is both orthogonal and positive definite. Indeed, if $K = K^T > 0$ is orthogonal, then $K^2 = \mathrm{I}$, and so its eigenvalues are all ± 1. Positive definiteness implies that all the eigenvalues must be $+1$, and hence its diagonal form is $\Lambda = \mathrm{I}$. But then $K = Q\, \mathrm{I}\, Q^T = \mathrm{I}$ also.

♢ 8.5.27. (*a*) Set $B = Q \sqrt{\Lambda}\, Q^T$, where $\sqrt{\Lambda}$ is the diagonal matrix with the square roots of the eigenvalues of A along the diagonal. Uniqueness follows from the fact that the eigenvectors and eigenvalues are uniquely determined. (Permuting them does not change the final form of B.) (*b*) (*i*) $\dfrac{1}{2}\begin{pmatrix} \sqrt{3}+1 & \sqrt{3}-1 \\ \sqrt{3}-1 & \sqrt{3}+1 \end{pmatrix}$; (*iii*) $\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

8.5.29. (*b*) $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & 3\sqrt{5} \end{pmatrix}$,

(*d*) $\begin{pmatrix} 0 & -3 & 8 \\ 1 & 0 & 0 \\ 0 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{5} & \frac{4}{5} \\ 1 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{pmatrix}$.

♡ 8.5.32. (*i*) This follows immediately from the spectral factorization. The rows of ΛQ^T are $\lambda_1 \mathbf{u}_1^T, \ldots, \lambda_n \mathbf{u}_n^T$, and formula (8.37) follows from the alternative version of matrix multiplication given in Exercise 1.2.34.

(*ii*) (*a*) $\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} = 5 \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} - 5 \begin{pmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}$.

8.5.33. Maximum: 7; minimum: 3.

8.5.35. (*a*) Maximum: 3; minimum: -2.

(*c*) Maximum: $\frac{8+\sqrt{5}}{2} = 5.11803$; minimum: $\frac{8-\sqrt{5}}{2} = 2.88197$.

8.5.37. (*a*) $\frac{5+\sqrt{5}}{2} = \max\{\, 2x^2 - 2xy + 3y^2 \mid x^2 + y^2 = 1 \,\}$,

$\frac{5-\sqrt{5}}{2} = \min\{\, 2x^2 - 2xy + 3y^2 \mid x^2 + y^2 = 1 \,\}$;

(*c*) $12 = \max\{\, 6x^2 - 8xy + 2xz + 6y^2 - 2yz + 11z^2 \mid x^2 + y^2 + z^2 = 1 \,\}$,

$2 = \min\{\, 6x^2 - 8xy + 2xz + 6y^2 - 2yz + 11z^2 \mid x^2 + y^2 + z^2 = 1 \,\}$

8.5.38. (*c*) $9 = \max\{\, 6x^2 - 8xy + 2xz + 6y^2 - 2yz + 11z^2 \mid x^2 + y^2 + z^2 = 1, x - y + 2z = 0 \,\}$.

♢ 8.5.42. According to the discussion preceding the statement of the Theorem 8.42,

$$\lambda_j = \max\left\{ \mathbf{y}^T \Lambda\, \mathbf{y} \;\middle|\; \|\mathbf{y}\| = 1,\ \mathbf{y}\cdot\mathbf{e}_1 = \cdots = \mathbf{y}\cdot\mathbf{e}_{j-1} = 0 \right\}.$$

Moreover, using (8.36), setting $\mathbf{x} = Q\,\mathbf{y}$ and using the fact that Q is an orthogonal matrix and so $(Q\,\mathbf{v})\cdot(Q\,\mathbf{w}) = \mathbf{v}\cdot\mathbf{w}$ for any $\mathbf{v},\mathbf{w}\in\mathbb{R}^n$, we have

$$\mathbf{x}^T A\,\mathbf{x} = \mathbf{y}^T \Lambda\,\mathbf{y}, \qquad \|\mathbf{x}\| = \|\mathbf{y}\|, \qquad \mathbf{y}\cdot\mathbf{e}_i = \mathbf{x}\cdot\mathbf{v}_i,$$

where $\mathbf{v}_i = Q\,\mathbf{e}_i$ is the i^{th} eigenvector of A. Therefore, by the preceding formula,

$$\lambda_j = \max\left\{ \mathbf{x}^T A\,\mathbf{x} \;\middle|\; \|\mathbf{x}\| = 1,\ \mathbf{x}\cdot\mathbf{v}_1 = \cdots = \mathbf{x}\cdot\mathbf{v}_{j-1} = 0 \right\}.$$

8.5.46. (*a*) Maximum: $\frac{3}{4}$; minimum: $\frac{2}{5}$. (*c*) Maximum: 2; minimum: $\frac{1}{2}$.

8.6.1. (*a*) $U = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}, \quad \Delta = \begin{pmatrix} 2 & -2 \\ 0 & 2 \end{pmatrix};$

(*c*) $U = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}, \quad \Delta = \begin{pmatrix} 2 & 15 \\ 0 & -1 \end{pmatrix};$

♢ 8.6.4. If A is symmetric, its eigenvalues are real, and hence its Schur Decomposition is $A = Q\,\Delta\,Q^T$, where Q is an orthogonal matrix. But $A^T = (QTQ^T)^T = QT^TQ^T$, and hence $\Delta^T = \Delta$ is a symmetric upper triangular matrix, which implies that $\Delta = \Lambda$ is a diagonal matrix with the eigenvalues of A along its diagonal.

♡ 8.6.5. (*a*) If A is real, $A^\dagger = A^T$, and so if $A = A^T$ then $A^TA = A^2 = A\,A^T$.

(*b*) If A is unitary, then $A^\dagger A = \mathrm{I} = A\,A^\dagger$.

(*c*) Every real orthogonal matrix is unitary, so this follows from part (*b*).

8.6.6. (*b*) Two 1×1 Jordan blocks; eigenvalues $-3, 6$; eigenvectors $\mathbf{e}_1, \mathbf{e}_2$.

(*d*) One 3×3 Jordan block; eigenvalue 0; eigenvector $\mathbf{e}_1$.

(*e*) One 1×1, 2×2, and 1×1 Jordan blocks; eigenvalues $4,3,2$; eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4$.

8.6.7. (*a*) Eigenvalue: 2; Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ \frac13 \end{pmatrix};$

Jordan canonical form: $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$

(*c*) Eigenvalue: 1; Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix};$

Jordan canonical form: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$

(*e*) Eigenvalues: $-2, 0$; Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$;

Jordan canonical form: $\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

8.6.8. $\begin{pmatrix} 2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2 \end{pmatrix}, \begin{pmatrix} 2&1&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2 \end{pmatrix}, \begin{pmatrix} 2&0&0&0\\0&2&1&0\\0&0&2&0\\0&0&0&2 \end{pmatrix}, \begin{pmatrix} 2&0&0&0\\0&2&0&0\\0&0&2&1\\0&0&0&2 \end{pmatrix},$
$\begin{pmatrix} 2&1&0&0\\0&2&0&0\\0&0&2&1\\0&0&0&2 \end{pmatrix}, \begin{pmatrix} 2&1&0&0\\0&2&1&0\\0&0&2&0\\0&0&0&2 \end{pmatrix}, \begin{pmatrix} 2&0&0&0\\0&2&1&0\\0&0&2&1\\0&0&0&2 \end{pmatrix}, \begin{pmatrix} 2&1&0&0\\0&2&1&0\\0&0&2&1\\0&0&0&2 \end{pmatrix}.$

8.6.11. True. All Jordan chains have length one, and so consist only of eigenvectors.

8.6.16. True. If $\mathbf{z}_j = c\,\mathbf{w}_j$, then $A\,\mathbf{z}_j = cA\,\mathbf{w}_j = c\,\lambda\mathbf{w}_j + c\,\mathbf{w}_{j-1} = \lambda\mathbf{z}_j + \mathbf{z}_{j-1}$.

♢ 8.6.19. (*a*) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\mathbf{e}_2$ is an eigenvector of $A^2 = \mathrm{O}$, but not an eigenvector of A.

(*b*) Suppose $A = S\,J\,S^{-1}$ where J is the Jordan canonical form of A.
Then $A^2 = S\,J^2 S^{-1}$. Now, even though J^2 is not necessarily a Jordan matrix, cf. Exercise 8.6.18, since J is upper triangular with the eigenvalues on the diagonal, J^2 is also upper triangular and its diagonal entries, which are its eigenvalues and the eigenvalues of A^2, are the squares of the diagonal entries of J.

♢ 8.6.24. First, since $J_{\lambda,n}$ is upper triangular, its eigenvalues are its diagonal entries, and hence λ is the only eigenvalue. Moreover, $\mathbf{v} = (\,v_1, v_2, \ldots, v_n\,)^T$ is an eigenvector if and only if $(J_{\lambda,n} - \lambda\,\mathrm{I}\,)\mathbf{v} = (\,v_2, \ldots, v_n, 0\,)^T = \mathbf{0}$. This requires $v_2 = \cdots = v_n = 0$, and hence $\mathbf{v}$ must be a scalar multiple of $\mathbf{e}_1$.

8.6.26. (a) $\{\mathbf{0}\}$, the y-axis, $\mathbb{R}^2$; (*c*) $\{\mathbf{0}\}$, the line spanned the eigenvector $(\,1, -2, 3\,)^T$, the plane spanned by $(\,1, -2, 3\,)^T, (\,0, 1, 0\,)^T$, and $\mathbb{R}^3$; (*e*) $\{\mathbf{0}\}$, the x-axis, the w-axis, the $x\,z$-plane, the $y\,w$-plane, $\mathbb{R}^4$.

8.7.1. (a) $\sqrt{3 \pm \sqrt{5}}$; (*c*) $5\sqrt{2}$; (*e*) $\sqrt{7}$, $\sqrt{2}$.

8.7.2. (a) $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} \\ \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{2}{\sqrt{10+2\sqrt{5}}} \end{pmatrix} \begin{pmatrix} \sqrt{3+\sqrt{5}} & 0 \\ 0 & \sqrt{3-\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{-2+\sqrt{5}}{\sqrt{10-4\sqrt{5}}} & \frac{1}{\sqrt{10-4\sqrt{5}}} \\ \frac{-2-\sqrt{5}}{\sqrt{10+4\sqrt{5}}} & \frac{1}{\sqrt{10+4\sqrt{5}}} \end{pmatrix},$

(*c*) $\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} (\,5\sqrt{2}\,) \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix},$

(*e*) $\begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} -\frac{4}{\sqrt{35}} & -\frac{3}{\sqrt{35}} & \frac{1}{\sqrt{35}} & \frac{3}{\sqrt{35}} \\ \frac{2}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}.$

8.7.4. (a) The eigenvalues of $K = A^T A$ are $\frac{15}{2} \pm \frac{\sqrt{221}}{2} = 14.933,\ .0667$. The square roots of these eigenvalues give us the singular values of A. i.e., 3.8643, .2588. The condition number is 3.86433 / .25878 = 14.9330.

(c) The singular values are 3.1624, .0007273, and so the condition number is 3.1624 / .0007273 = 4348.17; the matrix is slightly ill-conditioned.

♠ 8.7.6. In all cases, the large condition number results in an inaccurate solution.

(a) The exact solution is $x = 1, y = -1$; with three digit rounding, the computed solution is $x = 1.56, y = -1.56$. The singular values of the coefficient matrix are 1615.22, .274885, and the condition number is 5876.

8.7.8. Let $A = \mathbf{v} \in \mathbb{R}^n$ be the matrix (column vector) in question. (a) It has one singular value: $\|\mathbf{v}\|$; (b) $P = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$, $\Sigma = \big(\|\mathbf{v}\|\big)$ — a 1×1 matrix, $Q = (1)$; (c) $\mathbf{v}^+ = \dfrac{\mathbf{v}^T}{\|\mathbf{v}\|^2}$.

8.7.10. Almost true, with but one exception — the zero matrix.

8.7.13. True. If $A = P\,\Sigma\,Q^T$ is the singular value decomposition of A, then the transposed equation $A^T = Q\,\Sigma\,P^T$ gives the singular value decomposition of A^T, and so the diagonal entries of Σ are also the singular values of A^T. Note that the square of the singular values of A are the nonzero eigenvalues of $K = A^T A$, whereas the square of the singular values of A^T are the nonzero eigenvalues of $\widetilde{K} = A A^T \neq K$. Thus, this result implies that the two Gram matrices have the same non-zero eigenvalues.

8.7.16. False. For example, $U = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ has singular values $3 \pm \sqrt{5}$.

8.7.21. False. For example, the 2×2 diagonal matrix with diagonal entries $2 \cdot 10^k$ and 10^{-k} for $k \gg 0$ has determinant 2 but condition number $2 \cdot 10^{2k}$.

8.7.26. (a) .671855, (c) .9755, (e) 1.1066.

8.7.29. (a) $\|A\| = \frac{7}{2}$. The "unit sphere" for this norm is the rectangle with corners $\left(\pm\frac{1}{2}, \pm\frac{1}{3}\right)^T$. It is mapped to the parallelogram with corners $\pm\left(\frac{5}{6}, -\frac{1}{6}\right)^T$, $\pm\left(\frac{1}{6}, \frac{7}{6}\right)^T$, with respective norms $\frac{5}{3}$ and $\frac{7}{2}$, and so $\|A\| = \max\{\,\|A\mathbf{v}\| \mid \|\mathbf{v}\| = 1\,\} = \frac{7}{2}$.

(c) According to Exercise 8.7.28, $\|A\|$ is the square root of the largest generalized eigenvalue of the matrix pair $K = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $A^T K A = \begin{pmatrix} 5 & -4 \\ -4 & 14 \end{pmatrix}$.

Thus, $\|A\| = \sqrt{\frac{43+\sqrt{553}}{12}} = 2.35436$.

8.7.33. (b) $\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}$, ($d$) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, ($f$) $\begin{pmatrix} \frac{1}{140} & \frac{1}{70} & \frac{3}{140} \\ \frac{3}{140} & \frac{3}{70} & \frac{9}{140} \end{pmatrix}$.

8.7.34. (b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}$, $A^+ = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{5}{14} \\ \frac{2}{7} & \frac{1}{14} \end{pmatrix}$, $\mathbf{x}^\star = A^+ \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{9}{7} \\ \frac{15}{7} \\ \frac{11}{7} \end{pmatrix}$.

8.7.38. We list the eigenvalues of the graph Laplacian; the singular values of the incidence matrix are obtained by taking square roots.

$$(i)\ \ 4, 3, 1, 0; \qquad (iii)\ \ \frac{7+\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}, \frac{7-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}, 0.$$

8.8.1. Assuming $\nu = 1$: (*b*) Mean $= 1.275$; variance $= 3.995$; standard deviation $= 1.99875$.
(*d*) Mean $= .4$; variance $= 2.36$; standard deviation $= 1.53623$.

8.8.2. Assuming $\nu = 1$: (*b*) Mean $= .36667$; variance $= 2.24327$; standard deviation $= 1.49775$.
(*d*) Mean $= 1.19365$; variance $= 10.2307$; standard deviation $= 3.19855$.

8.8.4. For this to be valid, we need to take $\nu = 1/m$. Then

$$\begin{aligned}\sigma_{xy} &= \frac{1}{m}\sum_{i=1}^{m}(x_i-\overline{x})(y_i-\overline{y}) \\ &= \frac{1}{m}\sum_{i=1}^{m} x_i y_i - \overline{x}\left(\frac{1}{m}\sum_{i=1}^{m} y_i\right) - \overline{y}\left(\frac{1}{m}\sum_{i=1}^{m} y_i\right) + \overline{x}\,\overline{y} = \overline{x\,y} - \overline{x}\,\overline{y}.\end{aligned}$$

8.8.6. Observe that the row vector containing the column sums of A is obtained by left multiplication by the row vector $\mathbf{e} = (1, \ldots, 1)$ containing all 1s. Thus the columns sums of A are all zero if and only if $\mathbf{e}A = \mathbf{0}$. But then clearly, $\mathbf{e}AB = \mathbf{0}$.

♣ 8.8.8. (*a*) The singular values/principal variances are 31.8966, .93037, .02938, .01335 with

$$\text{principal directions}\quad \begin{pmatrix} .08677 \\ -.34555 \\ .77916 \\ -.27110 \\ -.43873 \end{pmatrix}, \begin{pmatrix} -.80181 \\ -.44715 \\ .08688 \\ -.05630 \\ .38267 \end{pmatrix}, \begin{pmatrix} .46356 \\ -.05729 \\ .31273 \\ -.18453 \\ .80621 \end{pmatrix}, \begin{pmatrix} -.06779 \\ .13724 \\ -.29037 \\ -.94302 \\ -.05448 \end{pmatrix}.$$

(*b*) The fact that there are only 4 nonzero singular values tells us that the data lies on a four-dimensional subspace. Moreover, the relative smallness of two smaller singular values indicates that the data can be viewed as a noisy representation of points on a two-dimensional subspace.

Students' Solutions Manual for

Chapter 9: Iteration

9.1.1. (*a*) $u^{(1)} = 2,\ u^{(10)} = 1024,\ u^{(20)} = 1048576$; unstable.
(*c*) $u^{(1)} = \mathrm{i}\,,\ u^{(10)} = -1,\ u^{(20)} = 1$; stable.

9.1.2. (*a*) $u^{(k+1)} = 1.0325\,u^{(k)}$, $u^{(0)} = 100$, where $u^{(k)}$ represents the balance after k years.
(*b*) $u^{(10)} = 1.0325^{10} \times 100 = 137.69$ dollars.
(*c*) $u^{(k+1)} = (1 + .0325/12)\,u^{(k)} = 1.002708\,u^{(k)}$, $\mathbf{u}^{(0)} = 100$, where $u^{(k)}$ represents the balance after k months. $u^{(120)} = (1 + .0325/12)^{120} \times 100 = 138.34$ dollars.

9.1.6. $|\,u^{(k)}\,| = |\,\lambda\,|^k\,|\,a\,| > |\,v^{(k)}\,| = |\,\mu\,|^k\,|\,b\,|$ provided $k > \dfrac{\log|\,b\,| - \log|\,a\,|}{\log|\,\lambda\,| - \log|\,\mu\,|}$, where the inequality relies on the fact that $\log|\,\lambda\,| > \log|\,\mu\,|$.

9.1.10. Let $u^{(k)}$ represent the balance after k years. Then $u^{(k+1)} = 1.05\,u^{(k)} + 120$, with $u^{(0)} = 0$. The equilibrium solution is $u^\star = -120/.05 = -2400$, and so after k years the balance is $u^{(k)} = (1.05^k - 1)\cdot 2400$. Then

$$u^{(10)} = \$1,509.35, \qquad u^{(50)} = \$25,121.76, \qquad u^{(200)} = \$4,149,979.40.$$

9.1.13. (*a*) $u^{(k)} = \dfrac{3^k + (-1)^k}{2}, \quad v^{(k)} = \dfrac{-3^k + (-1)^k}{2}$;
(*c*) $u^{(k)} = \dfrac{(\sqrt5+2)(3-\sqrt5)^k + (\sqrt5-2)(3+\sqrt5)^k}{2\sqrt5}, \qquad v^{(k)} = \dfrac{(3-\sqrt5)^k - (3+\sqrt5)^k}{2\sqrt5}$.

9.1.14. (*a*) $\mathbf{u}^{(k)} = c_1\,(-1-\sqrt2)^k \begin{pmatrix} -\sqrt2 \\ 1 \end{pmatrix} + c_2\,(-1+\sqrt2)^k \begin{pmatrix} \sqrt2 \\ 1 \end{pmatrix}$,

(*b*)
$$\mathbf{u}^{(k)} = c_1 \left(\tfrac12 + \tfrac{\sqrt3}{2}\,\mathrm{i}\right)^k \begin{pmatrix} \frac{5-\mathrm{i}\sqrt3}{2} \\ 1 \end{pmatrix} + c_2 \left(\tfrac12 - \tfrac{\sqrt3}{2}\,\mathrm{i}\right)^k \begin{pmatrix} \frac{5+\mathrm{i}\sqrt3}{2} \\ 1 \end{pmatrix} .$$
$$= a_1 \begin{pmatrix} \frac52 \cos\frac13 k\pi + \frac{\sqrt3}{2}\sin\frac13 k\pi \\ \cos\frac13 k\pi \end{pmatrix} + a_2 \begin{pmatrix} \frac52 \sin\frac13 k\pi - \frac{\sqrt3}{2}\cos\frac13 k\pi \\ \sin\frac13 k\pi \end{pmatrix}$$

9.1.16. (*a*) It suffices to note that the Lucas numbers are the general Fibonacci numbers (9.16) when $a = L^{(0)} = 2,\ b = L^{(1)} = 1$. (*b*) $2, 1, 3, 4, 7, 11, 18$.

9.1.18. (*b*) $\begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}^k = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3^k & 0 \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$,

(*d*)
$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4^k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^k \end{pmatrix} \begin{pmatrix} \frac13 & \frac13 & \frac13 \\ \frac16 & -\frac13 & \frac16 \\ -\frac12 & 0 & \frac12 \end{pmatrix}.$$

9.1.19. (b) $\begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3^k \\ -2^{k+1} \end{pmatrix}$, (d) $\begin{pmatrix} u^{(k)} \\ v^{(k)} \\ w^{(k)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{3}\, 4^k \\ \frac{1}{3} \\ 0 \end{pmatrix}$.

9.1.22. (a) $u^{(k)} = \frac{4}{3} - \frac{1}{3}(-2)^k$, (c) $u^{(k)} = \dfrac{(5-3\sqrt{5})(2+\sqrt{5})^k + (5+3\sqrt{5})(2-\sqrt{5})^k}{10}$.

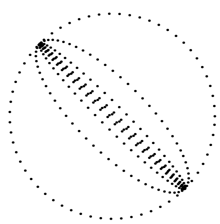
♣ 9.1.26. (a) $u^{(k)} = u^{(k-1)} + u^{(k-2)} - u^{(k-8)}$.

(b) $0, 1, 1, 2, 3, 5, 8, 13, 21, 33, 53, 84, 134, 213, 339, 539, 857, 1363, 2167, \ldots$.

(c) $\mathbf{u}^{(k)} = \left(u^{(k)}, u^{(k+1)}, \ldots, u^{(k+7)} \right)^T$ satisfies $\mathbf{u}^{(k+1)} = A\,\mathbf{u}^{(k)}$ where the 8×8 coefficient matrix A has 1's on the superdiagonal, last row $(-1, 0, 0, 0, 0, 0, 1, 1)$ and all other entries 0.

(d) The growth rate is given by largest eigenvalue in magnitude: $\lambda_1 = 1.59$, with $u^{(k)} \propto 1.59^k$. For more details, see [**44**].

♠ 9.1.29. (a)



E_1: principal axes: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; semi-axes: $1, \frac{1}{3}$; area: $\frac{1}{3}\pi$.

E_2: principal axes: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; semi-axes: $1, \frac{1}{9}$; area: $\frac{1}{9}\pi$.

E_3: principal axes: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; semi-axes: $1, \frac{1}{27}$; area: $\frac{1}{27}\pi$.

E_4: principal axes: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; semi-axes: $1, \frac{1}{81}$; area: $\frac{1}{81}\pi$.

9.1.32. $\mathbf{v}^{(k)} = c_1 (\alpha\lambda_1 + \beta)^k \mathbf{v}_1 + \cdots + c_n (\alpha\lambda_n + \beta)^k \mathbf{v}_n$.

9.1.35. According to Theorem 8.32, the eigenvectors of T are real and form an orthogonal basis of $\mathbb{R}^n$ with respect to the Euclidean norm. The formula for the coefficients c_j thus follows directly from (4.8).

9.1.37. Separating the equation into its real and imaginary parts, we find

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix}.$$

The eigenvalues of the coefficient matrix are $\mu \pm \mathrm{i}\,\nu$, with eigenvectors $\begin{pmatrix} 1 \\ \mp\mathrm{i} \end{pmatrix}$ and so the solution is

$$\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} = \frac{x^{(0)} + \mathrm{i}\,y^{(0)}}{2} (\mu + \mathrm{i}\,\nu)^k \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} + \frac{x^{(0)} - \mathrm{i}\,y^{(0)}}{2} (\mu - \mathrm{i}\,\nu)^k \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}.$$

Therefore $z^{(k)} = x^{(k)} + \mathrm{i}\,y^{(k)} = (x^{(0)} + \mathrm{i}\,y^{(0)})(\mu + \mathrm{i}\,\nu)^k = \lambda^k z^{(0)}$.

9.1.41. (a) $u^{(k)} = 2^k\left(c_1 + \frac{1}{2}\,k\,c_2\right),\ \ v^{(k)} = \frac{1}{3}\,2^k\,c_2;$

(c) $u^{(k)} = (-1)^k\left(c_1 - k\,c_2 + \frac{1}{2}\,k(k-1)\,c_3\right),\ v^{(k)} = (-1)^k\left(c_2 - (k+1)\,c_3\right),\ w^{(k)} = (-1)^k\,c_3.$

♡ 9.1.43. (a) The system has an equilibrium solution if and only if $(T - \mathrm{I}\,)\mathbf{u}^\star = \mathbf{b}$. In particular, if 1 is not an eigenvalue of T, every $\mathbf{b}$ leads to an equilibrium solution.

(b) Since $\mathbf{v}^{(k+1)} = T\,\mathbf{v}^{(k)}$, the general solution is

$$\mathbf{u}^{(k)} = \mathbf{u}^\star + c_1\,\lambda_1^k\,\mathbf{v}_1 + c_2\,\lambda_2^k\,\mathbf{v}_2 + \ \cdots\ + c_n\,\lambda_n^k\,\mathbf{v}_n,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ the corresponding eigenvalues of T.

(c) (i) $\mathbf{u}^{(k)} = \begin{pmatrix} \frac{2}{3} \\ -1 \end{pmatrix} - 5^k \begin{pmatrix} -\,3 \\ 1 \end{pmatrix} - (-3)^k \begin{pmatrix} -\,\frac{1}{3} \\ 1 \end{pmatrix};$

(iii) $\mathbf{u}^{(k)} = \begin{pmatrix} -1 \\ -\frac{3}{2} \\ -1 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \frac{15}{2}\,(-2)^k \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} - 5\,(-3)^k \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}.$

9.2.1. (a) Eigenvalues: $\frac{5+\sqrt{33}}{2} \simeq 5.3723,\ \ \frac{5-\sqrt{33}}{2} \simeq -\,.3723$; spectral radius: $\frac{5+\sqrt{33}}{2} \simeq 5.3723$.

(d) Eigenvalues: $4, -1 \pm 4\,\mathrm{i}$; spectral radius: $\sqrt{17} \simeq 4.1231$.

9.2.2. (a) Eigenvalues: $2 \pm 3\,\mathrm{i}$; spectral radius: $\sqrt{13} \simeq 3.6056$; not convergent.

(c) Eigenvalues: $\frac{4}{5}, \frac{3}{5}, 0$; spectral radius: $\frac{4}{5}$; convergent.

9.2.3. (b) Unstable: eigenvalues $\frac{5+\sqrt{73}}{12} \simeq 1.12867,\ \ \frac{5-\sqrt{73}}{12} \simeq -\,.29533$;

(d) stable: eigenvalues $-1, \pm\,\mathrm{i}$; (e) unstable: eigenvalues $\frac{5}{4}, \frac{1}{4}, \frac{1}{4}$.

9.2.6. A solution $\mathbf{u}^{(k)} \to \mathbf{0}$ if and only if the initial vector $\mathbf{u}^{(0)} = c_1\,\mathbf{v}_1 + \cdots + c_j\,\mathbf{v}_j$ is a linear combination of the eigenvectors (or more generally, Jordan chain vectors) corresponding to eigenvalues satisfying $|\,\lambda_i\,| < 1$ for $i = 1, \ldots, j$.

9.2.10. Since $\rho(c\,A) = |\,c\,|\,\rho(A)$, then $c\,A$ is convergent if and only if $|\,c\,| < 1/\rho(A)$. So, technically, there isn't a largest c.

9.2.14. (a) False: $\rho(c\,A) = |\,c\,|\,\rho(A)$.

(c) True, since the eigenvalues of A^2 are the squares of the eigenvalues of A.

9.2.16. (a) $P^2 = \left(\lim_{k\to\infty} T^k\right)^2 = \lim_{k\to\infty} T^{2\,k} = P$. (b) The only eigenvalues of P are 1 and 0. Moreover, P must be complete, since if $\mathbf{v}_1, \mathbf{v}_2$ are the first two vectors in a Jordan chain, then $P\,v_1 = \lambda\,\mathbf{v}_1$, $P\,v_2 = \lambda\,\mathbf{v}_2 + \mathbf{v}_1$, with $\lambda = 0$ or 1, but $P^2\mathbf{v}_2 = \lambda^2\mathbf{v}_1 + 2\,\lambda\,\mathbf{v}_2 \neq P\,v_2 = \lambda\,\mathbf{v}_2 + \mathbf{v}_1$, so there are no Jordan chains except for the ordinary eigenvectors. Therefore, $P = S\,\mathrm{diag}\,(1, \ldots, 1, 0, \ldots 0)\,S^{-1}$ for some nonsingular matrix S.

9.2.23. (a) All scalar multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; (c) all scalar multiples of $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$.

9.2.24. (a) The eigenvalues are $1, \frac{1}{2}$, so the fixed points are stable, while all other solutions go to a unique fixed point at rate $\left(\frac{1}{2}\right)^k$. When $\mathbf{u}^{(0)} = (1, 0)^T$, then $\mathbf{u}^{(k)} \to \left(\frac{3}{5}, \frac{3}{5}\right)^T$.

(c) The eigenvalues are $-2, 1, 0$, so the fixed points are unstable. Most solutions, specifically those with a nonzero component in the dominant eigenvector direction, become unbounded. However, when $\mathbf{u}^{(0)} = (1, 0, 0)^T$, then $\mathbf{u}^{(k)} = (-1, -2, 1)^T$ for $k \geq 1$, and the solution stays at a fixed point.

9.2.26. False: T has an eigenvalue of 1, but convergence requires that all eigenvalues be less than 1 in modulus.

9.2.30. (a) $\frac{3}{4}$, convergent; (c) $\frac{8}{7}$, inconclusive; (e) $\frac{8}{7}$, inconclusive; (f) .9, convergent.

9.2.31. (a) .671855, convergent; (c) .9755, convergent; (e) 1.1066, inconclusive.

9.2.32. (a) $\frac{2}{3}$, convergent; (c) .9755, convergent; (e) .9437, convergent.

9.2.34. Since $\| c A \| = |c| \, \| A \| < 1$.

♢ 9.2.37. This follows directly from the fact, proved in Proposition 8.62, that the singular values of a symmetric matrix are just the absolute values of its nonzero eigenvalues.

9.2.41. For instance, any diagonal matrix whose diagonal entries satisfy $0 < |a_{ii}| < 1$.

9.2.42. (a) False: For instance, if $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, then $B = S^{-1} A S = \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix}$, and $\| B \|_\infty = 2 \neq 1 = \| A \|_\infty$. (c) True, since A and B have the same eigenvalues.

9.3.1. (b) Not a transition matrix; (d) regular transition matrix: $\left(\frac{1}{6}, \frac{5}{6}\right)^T$;

(e) not a regular transition matrix; (f) regular transition matrix: $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$.

9.3.4. 2004: 37,000 city, 23,000 country; 2005: 38,600 city, 21,400 country; 2006: 39,880 city, 20,120 country; 2007: 40,904 city, 19,096 country; 2008: 41,723 city, 18,277 country; Eventual: 45,000 in the city and 15,000 in the country.

9.3.7. 58.33% of the nights.

9.3.8. When in Atlanta he always goes to Boston; when in Boston he has a 50% probability of going to either Atlanta or Chicago; when in Chicago he has a 50% probability of going to either Atlanta or Boston. The transition matrix is regular because

$$T^4 = \begin{pmatrix} .375 & .3125 & .3125 \\ .25 & .5625 & .5 \\ .375 & .125 & .1875 \end{pmatrix} \qquad \text{has all positive entries.}$$

On average he visits Atlanta: 33.33%, Boston 44.44%, and Chicago: 22.22% of the time.

9.3.10. Numbering the vertices from top to bottom and left to right, the transition matrix is

$$T = \begin{pmatrix} 0 & \frac14 & \frac14 & 0 & 0 & 0 \\ \frac12 & 0 & \frac14 & \frac12 & \frac14 & 0 \\ \frac12 & \frac14 & 0 & 0 & \frac14 & \frac12 \\ 0 & \frac14 & 0 & 0 & \frac14 & 0 \\ 0 & \frac14 & \frac14 & \frac12 & 0 & \frac12 \\ 0 & 0 & \frac14 & 0 & \frac14 & 0 \end{pmatrix}. \text{ The probability eigenvector is } \begin{pmatrix} \frac19 \\ \frac29 \\ \frac29 \\ \frac19 \\ \frac29 \\ \frac19 \end{pmatrix} \text{ and so the bug spends,}$$

on average, twice as much time at the edge vertices as at the corner vertices.

9.3.14. The limit is $\begin{pmatrix} \frac13 & \frac13 & \frac13 \\ \frac13 & \frac13 & \frac13 \\ \frac13 & \frac13 & \frac13 \end{pmatrix}$.

9.3.19. All equal probabilities: $\mathbf{z} = \left(\frac1n, \ \ldots \ , \frac1n \right)^T$.

9.3.22. True. If $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ is a probability eigenvector, then $\displaystyle\sum_{i=1}^n v_i = 1$ and $\displaystyle\sum_{j=1}^n t_{ij} v_j = \lambda v_i$ for all $i = 1, \ldots, n$. Summing the latter equations over i, we find

$$\lambda = \lambda \sum_{i=1}^n v_i = \sum_{i=1}^n \sum_{j=1}^n t_{ij} v_j = \sum_{j=1}^n v_j = 1,$$

since the column sums of a transition matrix are all equal to 1.

♢ 9.3.24. The i^{th} entry of $\mathbf{v}$ is $v_i = \displaystyle\sum_{j=1}^n t_{ij} u_j$. Since each $t_{ij} \geq 0$ and $u_j \geq 0$, the sum $v_i \geq 0$ also. Moreover, $\displaystyle\sum_{i=1}^n v_i = \sum_{i,j=1}^n t_{ij} u_j = \sum_{j=1}^n u_j = 1$ because all the column sums of T are equal to 1, and $\mathbf{u}$ is a probability vector.

9.4.1. (a) The eigenvalues are $-\frac12, \frac13$, so $\rho(T) = \frac12$.

(b) The iterates will converge to the fixed point $\left(-\frac16, 1\right)^T$ at rate $\frac12$. Asymptotically, they come in to the fixed point along the direction of the dominant eigenvector $(-3, 2)^T$.

9.4.2. (a) $\rho(T) = 2$; the iterates diverge: $\| \mathbf{u}^{(k)} \| \to \infty$ at a rate of 2.

9.4.3. (a) Strictly diagonally dominant; (c) not strictly diagonally dominant;
(e) strictly diagonally dominant.

♠ 9.4.4. (a) $x = \frac17 = .142857$, $y = -\frac27 = -.285714$;
(e) $x = -1.9172$, $y = -.339703$, $z = -2.24204$.

♠ 9.4.5. (c) Jacobi spectral radius $= .547723$, so Jacobi converges to the solution
$x = \frac87 = 1.142857$, $y = \frac{19}{7} = 2.71429$.

9.4.6. (a) $\mathbf{u} = \begin{pmatrix} .7857 \\ .3571 \end{pmatrix}$, (c) $\mathbf{u} = \begin{pmatrix} .3333 \\ -1.0000 \\ 1.3333 \end{pmatrix}$.

9.4.8. If $A\mathbf{u} = \mathbf{0}$, then $D\,\mathbf{u} = -\,(L+U)\,\mathbf{u}$, and hence $T\,\mathbf{u} = -\,D^{-1}(L+U)\,\mathbf{u} = \mathbf{u}$, proving that $\mathbf{u}$ is a eigenvector for T with eigenvalue 1. Therefore, $\rho(T) \geq 1$, which implies that T is *not* a convergent matrix.

9.4.11. False for elementary row operations of types 1 & 2, but true for those of type 3.

♠ 9.4.13. (a) $x = \frac{1}{7} = .142857$, $y = -\frac{2}{7} = -.285714$;
(e) $x = -1.9172$, $y = -.339703$, $z = -2.24204$.

9.4.14. (a) $\rho_J = .2582$, $\rho_{GS} = .0667$; Gauss–Seidel converges faster.
(c) $\rho_J = .5477$, $\rho_{GS} = .3$; Gauss–Seidel converges faster.
(e) $\rho_J = .4541$, $\rho_{GS} = .2887$; Gauss–Seidel converges faster.

♠ 9.4.17. The solution is $x = .083799$, $y = .21648$, $z = 1.21508$. The Jacobi spectral radius is .8166, and so it converges reasonably rapidly to the solution; indeed, after 50 iterations, $x^{(50)} = .0838107$, $y^{(50)} = .216476$, $z^{(50)} = 1.21514$. On the other hand, the Gauss–Seidel spectral radius is 1.0994, and it slowly diverges; after 50 iterations, $x^{(50)} = -30.5295$, $y^{(50)} = 9.07764$, $z^{(50)} = -90.8959$.

♣ 9.4.22. (a) Diagonal dominance requires $|\,z\,| > 4$; (b) The solution is $\mathbf{u} = (.0115385,$ $-.0294314, -.0755853, .0536789, .31505, .0541806, -.0767559, -.032107, .0140468, .0115385)^T$. It takes 41 Jacobi iterations and 6 Gauss–Seidel iterations to compute the first three decimal places of the solution. (c) Computing the spectral radius, we conclude that the Jacobi Method converges to the solution whenever $|\,z\,| > 3.6387$, while the Gauss–Seidel Method converges for $z < -3.6386$ or $z > 2$.

♡ 9.4.24. (a) $\mathbf{u} = \begin{pmatrix} 1.4 \\ .2 \end{pmatrix}$.

(b) The spectral radius is $\rho_J = .40825$ and so it takes about $-1/\log_{10}\rho_J \simeq 2.57$ iterations to produce each additional decimal place of accuracy.

(c) The spectral radius is $\rho_{GS} = .16667$ and so it takes about $-1/\log_{10}\rho_{GS} \simeq 1.29$ iterations to produce each additional decimal place of accuracy.

(d) $\mathbf{u}^{(n+1)} = \begin{pmatrix} 1-\omega & -\frac{1}{2}\,\omega \\ -\frac{1}{3}\,(1-\omega)\,\omega & \frac{1}{6}\,\omega^2 - \omega + 1 \end{pmatrix} \mathbf{u}^{(n)} + \begin{pmatrix} \frac{3}{2}\,\omega \\ \frac{2}{3}\,\omega - \frac{1}{2}\,\omega^2 \end{pmatrix}$.

(e) The SOR spectral radius is minimized when the two eigenvalues of T_ω coincide, which occurs when $\omega_\star = 1.04555$, at which value $\rho_\star = \omega_\star - 1 = .04555$, so the optimal SOR Method is almost 3.5 times as fast as Jacobi, and about 1.7 times as fast as Gauss–Seidel.

(f) For Jacobi, about $-5/\log_{10}\rho_J \simeq 13$ iterations; for Gauss–Seidel, about $-5/\log_{10}\rho_{GS} =$ 7 iterations; for optimal SOR, about $-5/\log_{10}\rho_{SOR} \simeq 4$ iterations.

(g) To obtain 5 decimal place accuracy, Jacobi requires 12 iterations, Gauss–Seidel requires 6 iterations, while optimal SOR requires 5 iterations.

♣ 9.4.27. (a) $x = .5,\ \ y = .75,\ \ z = .25,\ \ w = .5$. (b) To obtain 5 decimal place accuracy, Jacobi requires 14 iterations, Gauss–Seidel requires 8 iterations. One can get very good approximations of the spectral radii $\rho_J = .5,\ \ \rho_{GS} = .25$, by taking ratios of entries of successive iterates, or the ratio of norms of successive error vectors. (c) The optimal SOR Method has $\omega = 1.0718$, and requires 6 iterations to get 5 decimal place accuracy. The SOR spectral radius is $\rho_{SOR} = .0718$.

♣ 9.4.31. The Jacobi spectral radius is $\rho_J = .909657$. Using equation (9.76) to fix the SOR parameter $\omega = 1.41307$ actually slows down the convergence since $\rho_{SOR} = .509584$ while $\rho_{GS} = .32373$. Computing the spectral radius directly, the optimal SOR parameter is $\omega_\star = 1.17157$ with $\rho_\star = .290435$. Thus, optimal SOR is about 13 times as fast as Jacobi, but only marginally faster than Gauss-Seidel.

♡ 9.4.35. (a)
$$\begin{aligned}\mathbf{u}^{(k+1)} &= \mathbf{u}^{(k)} + D^{-1}\mathbf{r}^{(k)} = \mathbf{u}^{(k)} - D^{-1}A\,\mathbf{u}^{(k)} + D^{-1}\mathbf{b}\\ &= \mathbf{u}^{(k)} - D^{-1}(L+D+U)\mathbf{u}^{(k)} + D^{-1}\mathbf{b} = -\,D^{-1}(L+U)\mathbf{u}^{(k)} + D^{-1}\mathbf{b},\end{aligned}$$
which agrees with (9.55).

♠ 9.5.1. In all cases, we use the normalized version (9.82) starting with $\mathbf{u}^{(0)} = \mathbf{e}_1$; the answers are correct to 4 decimal places. (a) After 17 iterations, $\lambda = 2.00002,\ \ \mathbf{u} = (\,-.55470, .83205\,)^T$.

(c) After 38 iterations, $\lambda = 3.99996,\ \ \mathbf{u} = (\,.57737, -.57735, .57734\,)^T$.

(e) After 36 iterations, $\lambda = 5.54911,\ \ \mathbf{u} = (\,-.39488, .71005, .58300\,)^T$.

♠ 9.5.2. In each case, to find the dominant singular value of a matrix A, we apply the Power Method to $K = A^T A$ and take the square root of its dominant eigenvalue to find the dominant singular value $\sigma_1 = \sqrt{\lambda_1}$ of A.

(a) $K = \begin{pmatrix} 2 & -1 \\ -1 & 13 \end{pmatrix}$; after 11 iterations, $\lambda_1 = 13.0902$ and $\sigma_1 = 3.6180$;

(c) $K = \begin{pmatrix} 5 & 2 & 2 & -1 \\ 2 & 8 & 2 & -4 \\ 2 & 2 & 1 & -1 \\ -1 & -4 & -1 & 2 \end{pmatrix}$; after 16 iterations, $\lambda_1 = 11.6055$ and $\sigma_1 = 3.4067$.

♢ 9.5.5. (a) If $A\mathbf{v} = \lambda\mathbf{v}$ then $A^{-1}\mathbf{v} = \dfrac{1}{\lambda}\,\mathbf{v}$, and so $\mathbf{v}$ is also the eigenvector of A^{-1}.

(b) If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, with $|\,\lambda_1\,| > |\,\lambda_2\,| > \cdots > |\,\lambda_n\,| > 0$ (recalling that 0 cannot be an eigenvalue if A is nonsingular), then $\dfrac{1}{\lambda_1}, \ldots, \dfrac{1}{\lambda_n}$ are the eigenvalues of A^{-1}, and $\dfrac{1}{|\,\lambda_n\,|} > \dfrac{1}{|\,\lambda_{n-1}\,|} > \cdots > \dfrac{1}{|\,\lambda_1\,|}$ and so $\dfrac{1}{\lambda_n}$ is the dominant eigenvalue of A^{-1}. Thus, applying the Power Method to A^{-1} will produce the reciprocal of the smallest (meaning the one closest to 0) eigenvalue of A and its corresponding eigenvector.

(c) The rate of convergence of the algorithm is the ratio $|\,\lambda_n/\lambda_{n-1}\,|$ of the moduli of the smallest two eigenvalues.

♠ 9.5.6. (a) After 15 iterations, we obtain $\lambda = .99998$, $\mathbf{u} = (\,.70711, -.70710\,)^T$.

(c) After 12 iterations, we obtain $\lambda = 1.00001$, $\mathbf{u} = (\,.40825, .81650, .40825\,)^T$.

(e) After 7 iterations, we obtain $\lambda = -.88536$, $\mathbf{u} = (\,-.88751, -.29939, .35027\,)^T$.

♠ 9.5.8. (a) After 11 iterations, we obtain $\nu^\star = 2.00002$, so $\lambda^\star = 1.0000$, $\mathbf{u} = (\,.70711, -.70710\,)^T$.

(c) After 10 iterations, $\nu^\star = 2.00000$, so $\lambda^\star = 1.00000$, $\mathbf{u} = (\,.40825, .81650, .40825\,)^T$.

9.5.11. (a) Eigenvalues: 6.7016, .2984; eigenvectors: $\begin{pmatrix} .3310 \\ .9436 \end{pmatrix}, \begin{pmatrix} .9436 \\ -.3310 \end{pmatrix}$.

(c) Eigenvalues: 4.7577, 1.9009, −1.6586; eigenvectors: $\begin{pmatrix} .2726 \\ .7519 \\ .6003 \end{pmatrix}, \begin{pmatrix} .9454 \\ -.0937 \\ -.3120 \end{pmatrix}, \begin{pmatrix} -.1784 \\ .6526 \\ -.7364 \end{pmatrix}$.

9.5.13. (a) Eigenvalues: 2, 1; eigenvectors: $\begin{pmatrix} -2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(c) Eigenvalues: 3.5842, −2.2899, 1.7057;

eigenvectors: $\begin{pmatrix} -.4466 \\ -.7076 \\ .5476 \end{pmatrix}, \begin{pmatrix} .1953 \\ -.8380 \\ -.5094 \end{pmatrix}, \begin{pmatrix} .7491 \\ -.2204 \\ .6247 \end{pmatrix}$.

9.5.15. (a) It has eigenvalues ± 1, which have the same magnitude. The $Q\,R$ factorization is trivial, with $Q = A$ and $R = \mathrm{I}$. Thus, $R\,Q = A$, and so nothing happens.

(b) It has a pair of complex conjugate eigenvalues of modulus $\sqrt{7}$ and a real eigenvalue -1. However, running the $Q\,R$ iteration produces a block upper triangular matrix with the real eigenvalue at position $(3,3)$ and a 2×2 upper left block that has the complex conjugate eigenvalues of A as eigenvalues.

9.5.18. (a)

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -.9615 & .2747 \\ 0 & .2747 & .9615 \end{pmatrix}, \qquad T = H\,A\,H = \begin{pmatrix} 8.0000 & 7.2801 & 0 \\ 7.2801 & 20.0189 & 3.5660 \\ 0 & 3.5660 & 4.9811 \end{pmatrix}.$$

♠ 9.5.19. (a) Eigenvalues: $24, 6, 3$.

9.5.21. (b)

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -.8944 & 0 & -.4472 \\ 0 & 0 & 1 & 0 \\ 0 & -.4472 & 0 & .8944 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 3.0000 & -2.2361 & -1.0000 & 0 \\ -2.2361 & 3.8000 & 2.2361 & .4000 \\ 0 & 1.7889 & 2.0000 & -5.8138 \\ 0 & 1.4000 & -4.4721 & 1.2000 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -.7875 & -.6163 \\ 0 & 0 & -.6163 & .7875 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3.0000 & -2.2361 & .7875 & .6163 \\ -2.2361 & 3.8000 & -2.0074 & -1.0631 \\ 0 & -2.2716 & -3.2961 & 2.2950 \\ 0 & 0 & .9534 & 6.4961 \end{pmatrix}.$$

♠ 9.5.22. (*b*) Eigenvalues: 7., 5.74606, −4.03877, 1.29271.

9.6.1. (*b*) $V^{(1)}: \begin{pmatrix} \frac{1}{\sqrt5} \\ -\frac{2}{\sqrt5} \\ 0 \end{pmatrix}, \qquad V^{(2)}: \begin{pmatrix} \frac{1}{\sqrt5} \\ -\frac{2}{\sqrt5} \\ 0 \end{pmatrix}, \qquad V^{(3)}: \begin{pmatrix} \frac{1}{\sqrt5} \\ -\frac{2}{\sqrt5} \\ 0 \end{pmatrix};$

(*d*) $V^{(1)}: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad V^{(2)}: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad V^{(3)}: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ t0 \end{pmatrix}.$

9.6.2. (*a*) Let $\lambda = \mu + \mathrm{i}\,\nu$. Then the eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$ implies $A\mathbf{x} = \mu\mathbf{x} - \nu\mathbf{y}$, $A\mathbf{y} = \nu\mathbf{x} + \mu\mathbf{y}$. Iterating, we see that every iterate $A^k\mathbf{x}$ and $A^k\mathbf{y}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$, and hence each $V^{(k)}$ is spanned by $\mathbf{x}, \mathbf{y}$ and hence two-dimensional, noting that $\mathbf{x}, \mathbf{y}$ are linearly independent according to Exercise 8.3.12(a).

9.6.7. In each case, the last $\mathbf{u}_k$ is the actual solution, with residual $\mathbf{r}_k = \mathbf{f} - K\mathbf{u}_k = \mathbf{0}$.

(*a*) $\mathbf{r}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ \mathbf{u}_1 = \begin{pmatrix} .76923 \\ .38462 \end{pmatrix}, \qquad \mathbf{r}_1 = \begin{pmatrix} .07692 \\ -.15385 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} .78571 \\ .35714 \end{pmatrix};$

(*c*) $\mathbf{r}_0 = \begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix}, \ \mathbf{u}_1 = \begin{pmatrix} -.13466 \\ -.26933 \\ .94264 \end{pmatrix}, \qquad \mathbf{r}_1 = \begin{pmatrix} 2.36658 \\ -4.01995 \\ -.81047 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} -.13466 \\ -.26933 \\ .94264 \end{pmatrix},$

$\mathbf{r}_2 = \begin{pmatrix} .72321 \\ .38287 \\ .21271 \end{pmatrix}, \ \mathbf{u}_3 = \begin{pmatrix} .33333 \\ -1.00000 \\ 1.33333 \end{pmatrix}.$

♣ 9.6.8. Remarkably, after only two iterations, the method finds the exact solution:

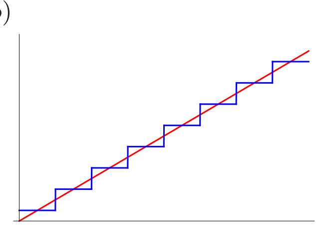
$$\mathbf{u}_3 = \mathbf{u}^\star = (\,.0625, .125, .0625, .125, .375, .125, .0625, .125, .0625\,)^T,$$

and hence the convergence is dramatically faster than the other iterative methods.

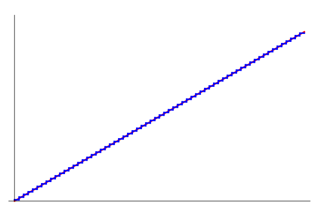
♢ 9.6.14. $d_k = \dfrac{\|\mathbf{r}_k\|^2}{\mathbf{r}_k^T A\mathbf{r}_k} = \dfrac{\mathbf{x}_k^T A^2\mathbf{x}_k - 2\,\mathbf{b}^T A\mathbf{x}_k + \|\mathbf{b}\|^2}{\mathbf{x}_k^T A^3\mathbf{x}_k - 2\,\mathbf{b}^T K^2\mathbf{x}_k + \mathbf{b}^T A\mathbf{b}}.$

♠ 9.7.1. (*a*) The coefficients are $c_0 = \frac12$ and $c_{j,k} = -2^{-j-2}$ for all $j \geq 0$ and $k = 0, \ldots, 2^j - 1$.

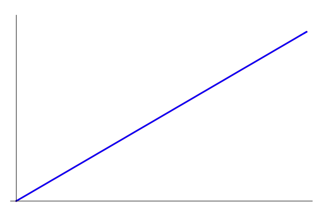
(*b*)



$r = 2$



$r = 5$



$r = 10$

For any integer $0 \le j \le 2^{r+1} - 1$, on the interval $j\,2^{-r-1} < x < (j+1)2^{-r-1}$, the value of the Haar approximant of order r is constant and equal to the value of the function $f(x) = x$ at the midpoint:

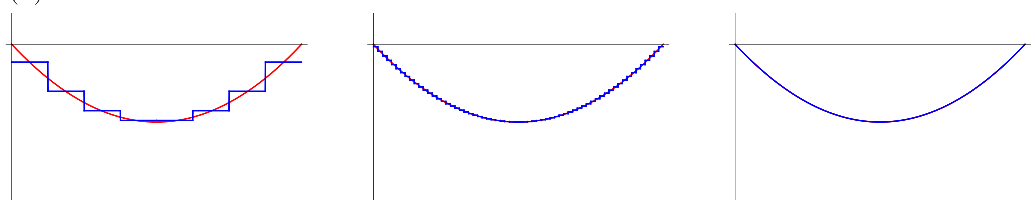
$$s_r(x) = f\Big((2j+1)\,2^{-r-2}\Big) = (2j+1)\,2^{-r-2}.$$

(Indeed, this is a general property of Haar wavelets.) Hence $|\,h_k(x) - f(x)\,| \le 2^{-r-2} \to 0$ as $r \to \infty$, proving convergence.

(*c*) Maximal deviation: $r = 2:$ .0625, $r = 5:$, .0078125 $r = 10:$ .0002441.

♠ 9.7.2. (*i*) (a) The coefficients are $c_0 = -\frac{1}{6}$, $c_{0,0} = 0$, $c_{j,k} = (2^j - 1 - 2k)\,2^{-2j-2}$ for $k = 0, \ldots, 2^j - 1$ and $j \ge 1$.

(*b*)



$r = 2$ $r = 5$ $r = 10$

In general, the Haar wavelet approximant $s_r(x)$ is constant on each subinterval $k\,2^{-r-1} < x < (k+1)\,2^{-r-1}$ for $k = 0, \ldots, 2^{r+1} - 1$ and equal to the value of the function $f(x)$ at the midpoint. This implies that the maximal error on each interval is bounded by the deviation of the function from its value at the midpoint, which suffices to prove convergence $s_r(x) \to f(x)$ at $r \to \infty$ provided $f(x)$ is continuous.

(*c*) Maximal deviation: $r = 2:$ .05729, $r = 5:$, .007731 $r = 10:$ .0002441.

♡ 9.7.4. (*b*)

$$W_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}, \qquad W_2^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{pmatrix}.$$

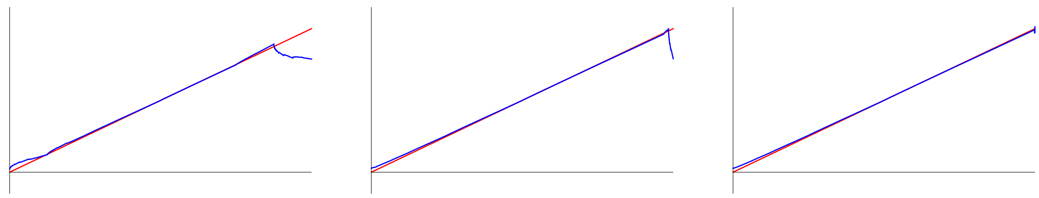
9.7.6. The function $\varphi(x) = \sigma(x) - \sigma(x-1)$ is merely a difference of two step functions — a *box function*. The mother wavelet is thus

$$w(x) = \varphi(2x) - \varphi(2x-1) = \sigma(x) - 2\,\sigma\Big(x - \tfrac{1}{2}\Big) + \sigma(x-1)$$

♠ 9.7.8.

Exercise 9.7.1: (a) The coefficients are $c_0 = .5181$, $c_{0,0} = -.1388$, $c_{1,0} = .05765$, $c_{1,1} = -.1833$, $c_{2,0} = c_{2,1} = 0$, $c_{2,2} = .05908$, $c_{2,3} = -.2055$, $c_{3,0} = c_{3,1} = c_{3,2} = c_{3,3} = c_{3,4} = c_{3,5} = -.01572$, $c_{3,6} = .05954$, $c_{3,7} = -.2167$.

(b)



$r = 2$ $\qquad$ $r = 5$ $\qquad$ $r = 10$

In general, on each interval $[\,k\,2^{-j}\,,k\,2^{-j}\,]$ for $k = 1, \dots, 02^j - 1$, the Haar wavelet approximant is constant, equal to the value of the function at the midpoint of the interval, and hence the error on each interval is bounded by the deviation of the function from the value at its midpoint. Thus, provided the function is continuous, convergence is guaranteed.

(c) Maximal deviation: $r = 2:\ .2116$, $\ r = 5:\ .2100$, $\ r = 10:\ .0331$. Note that the deviation is only of the indicated size near the end of the interval owing to the jump discontinuity at $x = 1$ and is much closer elsewhere.

♢ 9.7.10. For the box function, in view of (9.125),

$$\varphi(2x) = \begin{cases} 1, & 0 < x \le \frac{1}{2}, \\ 0, & \text{otherwise}, \end{cases} \qquad\qquad \varphi(2x-1) = \begin{cases} 1, & \frac{1}{2} < x \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

hence

$$\varphi(x) = \begin{cases} 1, & 0 < x \le 1, \\ 0, & \text{otherwise}, \end{cases} = \varphi(2x) + \varphi(2x-1),$$

proving (9.139).

♢ 9.7.12. We compute

$$\psi(x) = \varphi(x+1) = \varphi(2\,(x+1)-1) + \varphi(2\,(x+1)-2) = \varphi(2x+1) + \varphi(2x) = \psi(2x) + \psi(2x-1),$$

9.7.15. Since the inner product integral is translation invariant, by (9.147),

$$\begin{aligned} \langle\,\varphi(x-l)\,,\varphi(x-m)\,\rangle &= \int_{-\infty}^{\infty} \varphi(x-l)\,\varphi(x-m)\,dx \\ &= \int_{-\infty}^{\infty} \varphi(x)\,\varphi(x+l-m)\,dx = \langle\,\varphi(x)\,,\varphi(x+l-m)\,\rangle = 0 \end{aligned}$$

provided $l \ne m$.

9.7.18. To four decimal places: (a) 1.8659, (b) 1.27434.

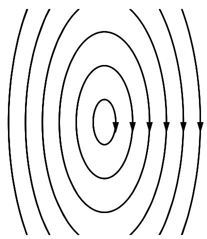
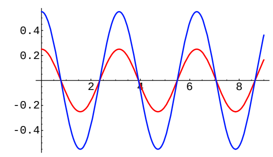
9.7.21. Almost true — the column sums of the coefficient matrix are both 1; however, the $(2,1)$ entry is negative, which is not an allowed probability in a Markov process.

9.7.25. (a) $2^{-1} + 2^{-2}$; $\quad$ (c) $1 + 2^{-2} + 2^{-3} + 2^{-5} + 2^{-7}$.

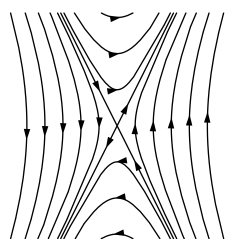
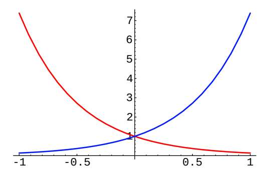
Students' Solutions Manual for
Chapter 10: Dynamics

10.1.1. (*i*) (a) $u(t) = c_1 \cos 2t + c_2 \sin 2t$. (*b*) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{u}$.

(*c*) $\mathbf{u}(t) = \begin{pmatrix} c_1 \cos 2t + c_2 \sin 2t \\ -2c_1 \sin 2t + 2c_2 \cos 2t \end{pmatrix}$.

(*d*)  (*e*) 

(*ii*) (a) $u(t) = c_1 e^{-2t} + c_2 e^{2t}$. (*b*) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \mathbf{u}$. (*c*) $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-2t} + c_2 e^{2t} \\ -2c_1 e^{-2t} + 2c_2 e^{2t} \end{pmatrix}$.

(*d*)  (*e*) 

◇ 10.1.4. (a) Use the chain rule to compute $\dfrac{d\mathbf{v}}{dt} = -\dfrac{d\mathbf{u}}{dt}(-t) = -A\,\mathbf{u}(-t) = -A\,\mathbf{v}$.

(*b*) Since $\mathbf{v}(t) = \mathbf{u}(-t)$ parameterizes the same curve as $\mathbf{u}(t)$, but in the reverse direction.

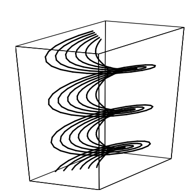
(*c*) (*i*) $\dfrac{d\mathbf{v}}{dt} = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \mathbf{v}$; solution: $\mathbf{v}(t) = \begin{pmatrix} c_1 \cos 2t - c_2 \sin 2t \\ 2c_1 \sin 2t + 2c_2 \cos 2t \end{pmatrix}$.

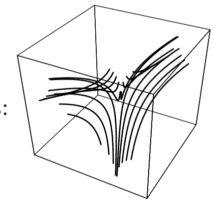
(*ii*) $\dfrac{d\mathbf{v}}{dt} = \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix} \mathbf{v}$; solution: $\mathbf{v}(t) = \begin{pmatrix} c_1 e^{2t} + c_2 e^{-2t} \\ -2c_1 e^{2t} + 2c_2 e^{-2t} \end{pmatrix}$.

10.1.8. False. If $\dot{\mathbf{u}} = A\,\mathbf{u}$ then the speed along the trajectory at the point $\mathbf{u}(t)$ is $\|\,A\,\mathbf{u}(t)\,\|$. So the speed is constant only if $\|\,A\,\mathbf{u}(t)\,\|$ is constant. (Later, in Lemma 10.29, this will be shown to correspond to A being a skew-symmetric matrix.)

♠ 10.1.10. In all cases, the t axis is plotted vertically, and the three-dimensional solution curves $(u(t), \dot{u}(t), t)^T$ project to the phase plane trajectories $(u(t), \dot{u}(t))^T$.

(*i*) The solution curves are helices going around the t axis:



(*ii*) Hyperbolic curves going away from the t axis in both directions: 

10.1.11. $u(t) = \frac{7}{5}\, e^{-5\,t} + \frac{8}{5}\, e^{5\,t}, \;\; v(t) = -\frac{14}{5}\, e^{-5\,t} + \frac{4}{5}\, e^{5\,t}.$

10.1.12. (*b*) $x_1(t) = -\,c_1\, e^{-5\,t} + 3\,c_2\, e^{5\,t}, \;\; x_2(t) = 3\,c_1\, e^{-5\,t} + c_2\, e^{5\,t};$

(*d*) $y_1(t) = -\,c_1\, e^{-t} - c_2\, e^{t} - \frac{2}{3}\, c_3, \;\; y_2(t) = c_1\, e^{-t} - c_2\, e^{t}, \;\; y_3(t) = c_1\, e^{-t} + c_2\, e^{t} + c_3.$

10.1.13. (a) $\mathbf{u}(t) = \left(\frac{1}{2}\, e^{2-2\,t} + \frac{1}{2}\, e^{-2+2\,t}, -\frac{1}{2}\, e^{2-2\,t} + \frac{1}{2}\, e^{-2+2\,t} \right)^T,$

(*c*) $\mathbf{u}(t) = \left(e^{t} \cos \sqrt{2}\, t, -\frac{1}{\sqrt{2}}\, e^{t} \sin \sqrt{2}\, t \right)^T,$

(*e*) $\mathbf{u}(t) = (\,-4 - 6\cos t - 9 \sin t, 2 + 3 \cos t + 6 \sin t, -1 - 3 \sin t\,)^T,$

10.1.15. $x(t) = e^{t/2}\left(\cos \frac{\sqrt{3}}{2}\, t - \sqrt{3}\, \sin \frac{\sqrt{3}}{2}\, t \right), \;\; y(t) = e^{t/2}\left(\cos \frac{\sqrt{3}}{2}\, t - \frac{1}{\sqrt{3}}\, \sin \frac{\sqrt{3}}{2}\, t \right),$
and so at time $t = 1$, the position is $(\, x(1), y(1)\,)^T = (\,-1.10719, .343028\,)^T$.

10.1.19. The general complex solution to the system is

$$\mathbf{u}(t) = c_1\, e^{-t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_2\, e^{(1+2\,\mathrm{i}\,)\,t} \begin{pmatrix} 1 \\ \mathrm{i} \\ 1 \end{pmatrix} + c_3\, e^{(1-2 i)\,t} \begin{pmatrix} 1 \\ -\,\mathrm{i} \\ 1 \end{pmatrix}.$$

Substituting into the initial conditions,

$$\mathbf{u}(0) = \begin{pmatrix} -\,c_1 + c_2 + c_3 \\ c_1 + \mathrm{i}\, c_2 - \,\mathrm{i}\, c_3 \\ c_1 + + c_2 + c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \qquad \text{we find} \qquad \begin{aligned} c_1 &= -\,2, \\ c_2 &= -\tfrac{1}{2}\,\mathrm{i}\,, \\ c_3 &= \tfrac{1}{2}\,\mathrm{i}\,. \end{aligned}$$

Thus, we obtain the same solution:

$$\mathbf{u}(t) = -\,2\, e^{-t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \tfrac{1}{2}\,\mathrm{i}\; e^{(1+2\,\mathrm{i}\,)\,t} \begin{pmatrix} 1 \\ \mathrm{i} \\ 1 \end{pmatrix} + \tfrac{1}{2}\,\mathrm{i}\; e^{(1-2 i)\,t} \begin{pmatrix} 1 \\ -\,\mathrm{i} \\ 1 \end{pmatrix} = \begin{pmatrix} 2\, e^{-t} + e^{t} \sin 2\,t \\ -\,2\, e^{-t} + e^{t} \cos 2\,t \\ -\,2\, e^{-t} + e^{t} \sin 2\,t \end{pmatrix}.$$

10.1.20. (*a*) Linearly independent; (*b*) linearly independent; (*d*) linearly dependent;
(*e*) linearly independent.

10.1.24. $\dfrac{d\mathbf{v}}{dt} = S\, \dfrac{d\mathbf{u}}{dt} = S\, A\, \mathbf{u} = S\, A\, S^{-1} \mathbf{v} = B\, \mathbf{v}.$

♢ 10.1.25. (*i*) This is an immediate consequence of the preceding two exercises.

(*ii*) (a) $\mathbf{u}(t) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1\, e^{-2\,t} \\ c_2\, e^{2\,t} \end{pmatrix}, \quad$ (*c*) $\mathbf{u}(t) = \begin{pmatrix} -\sqrt{2}\;\mathrm{i} & \sqrt{2}\;\mathrm{i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1\, e^{(1+\,\mathrm{i}\,\sqrt{2})\,t} \\ c_2\, e^{(1-\,\mathrm{i}\,\sqrt{2})t} \end{pmatrix},$

(*e*) $\mathbf{u}(t) = \begin{pmatrix} 4 & 3 + 2\,\mathrm{i} & 3 - 2\,\mathrm{i} \\ -2 & -2 - \,\mathrm{i} & -2 + \,\mathrm{i} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2\, e^{\,\mathrm{i}\,t} \\ c_3\, e^{-\,\mathrm{i}\,t} \end{pmatrix}.$

10.1.26. (a) $\begin{pmatrix} c_1 e^{2t} + c_2 t\, e^{2t} \\ c_2 e^{2t} \end{pmatrix}$, (c) $\begin{pmatrix} c_1 e^{-3t} + c_2\left(\frac{1}{2}+t\right) e^{-3t} \\ 2c_1 e^{-3t} + 2c_2 t\, e^{-3t} \end{pmatrix}$,

(e) $\begin{pmatrix} c_1 e^{-3t} + c_2 t\, e^{-3t} + c_3\left(1+\frac{1}{2}t^2\right) e^{-3t} \\ c_2\, e^{-3t} + c_3 t\, e^{-3t} \\ c_1 e^{-3t} + c_2 t e^{-3t} + \frac{1}{2} c_3 t^2 e^{-3t} \end{pmatrix}$.

10.1.27. (a) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 2 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \mathbf{u}$, (c) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{u}$, (e) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 2 & 3 & 0 \end{pmatrix} \mathbf{u}$.

10.1.28. (a) No, since neither $\dfrac{d\mathbf{u}_i}{dt}$, $i = 1, 2$, is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$. Or note that the trajectories described by the functions cross, violating uniqueness.

(b) No, since polynomial solutions a two-dimensional system can be at most first order in t.

(d) Yes: $\dot{\mathbf{u}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{u}$. (e) Yes: $\dot{\mathbf{u}} = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \mathbf{u}$.

♢ 10.1.32. (a) By direct computation,

$$\frac{d\mathbf{u}_j}{dt} = \lambda e^{\lambda t} \sum_{i=1}^{j} \frac{t^{j-i}}{(j-i)!}\, \mathbf{w}_i + e^{\lambda t} \sum_{i=1}^{j-1} \frac{t^{j-i-1}}{(j-i-1)!}\, \mathbf{w}_i,$$

which equals

$$A\,\mathbf{u}_j = e^{\lambda t} \sum_{i=1}^{j} \frac{t^{j-i}}{(j-i)!}\, A\,\mathbf{w}_i = e^{\lambda t} \left[\frac{t^{j-1}}{(j-1)!}\, \mathbf{w}_1 + \sum_{i=2}^{j} \frac{t^{j-i}}{(j-i)!}\, (\lambda\,\mathbf{w}_i + \mathbf{w}_{i-1}) \right].$$

(b) At $t = 0$, we have $\mathbf{u}_j(0) = \mathbf{w}_j$, and the Jordan chain vectors are linearly independent.

10.2.1. (a) Asymptotically stable: the eigenvalues are $-2 \pm \mathrm{i}$;
(d) stable: the eigenvalues are $\pm 4\,\mathrm{i}$; (f) unstable: the eigenvalues are $1, -1 \pm 2\,\mathrm{i}$.

10.2.3. (a) $\dot{u} = -2u$, $\dot{v} = -2v$, with solution $u(t) = c_1 e^{-2t}$, $v(t) = c_2 e^{-2t}$.
(c) $\dot{u} = -8u + 2v$, $\dot{v} = 2u - 2v$, with solution
$u(t) = -c_1 \frac{\sqrt{13}+3}{2}\, e^{-(5+\sqrt{13})t} + c_2 \frac{\sqrt{13}-3}{2}\, e^{-(5-\sqrt{13})t}$, $v(t) = c_1 e^{-(5+\sqrt{13})t} + c_2 e^{-(5-\sqrt{13})t}$.

10.2.5. (a) Gradient flow; asymptotically stable. (b) Neither; unstable.
(d) Hamiltonian flow; stable.

10.2.9. The system is stable since $\pm\,\mathrm{i}$ must be simple eigenvalues. Indeed, any 5×5 matrix has 5 eigenvalues, counting multiplicities, and the multiplicities of complex conjugate eigenvalues are the same. A 6×6 matrix can have $\pm\,\mathrm{i}$ as complex conjugate, incomplete double eigenvalues, in addition to the simple real eigenvalues $-1, -2$, and in such a situation the origin would be unstable.

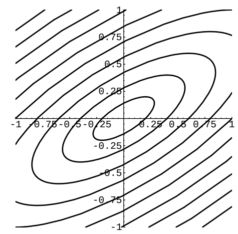
10.2.14. (a) True, since the sum of the eigenvalues equals the trace, so at least one must be positive or have positive real part in order that the trace be positive.

10.2.20. False. Only positive definite Hamiltonian functions lead to stable gradient flows.

♡ 10.2.22. (a) By the multivariable calculus chain rule

$$\frac{d}{dt}\,H(u(t),v(t)) = \frac{\partial H}{\partial u}\,\frac{du}{dt} + \frac{\partial H}{\partial v}\,\frac{dv}{dt} = \frac{\partial H}{\partial u}\,\frac{\partial H}{\partial v} + \frac{\partial H}{\partial v}\left(-\,\frac{\partial H}{\partial u}\right) \equiv 0.$$

Therefore $H(u(t),v(t)) \equiv c$ is constant, with its value $c = H(u_0,v_0)$ fixed by the initial conditions $u(t_0) = u_0, v(t_0) = v_0$.



(b) The solutions are

$$u(t) = c_1\cos(2t) - c_1\sin(2t) + 2\,c_2\sin(2t),$$
$$v(t) = c_2\cos(2t) - c_1\sin(2t) + c_2\sin(2t),$$

and leave the Hamiltonian function constant:

$$H(u(t),v(t)) = u(t)^2 - 2\,u(t)\,v(t) + 2\,v(t)^2 = c_1^2 - 2\,c_1\,c_2 + 2\,c_2^2 = c.$$

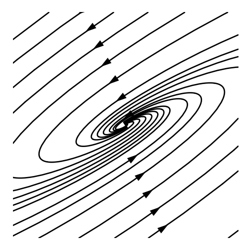
10.2.24. (a) The equilibrium solution satisfies $A\,\mathbf{u}^\star = -\,\mathbf{b}$, and so $\mathbf{v}(t) = \mathbf{u}(t) - \mathbf{u}^\star$ satisfies

$$\dot{\mathbf{v}} = \dot{\mathbf{u}} = A\,\mathbf{u} + \mathbf{b} = A(\mathbf{u} - \mathbf{u}^\star) = A\,\mathbf{v},$$

which is the homogeneous system.

(b) (i) $u(t) = -\,3\,c_1\,e^{2\,t} + c_2\,e^{-\,2\,t} - \frac{1}{4}, \quad v(t) = c_1\,e^{2\,t} + c_2\,e^{-\,2\,t} + \frac{1}{4}.$

10.3.1. (ii) $A = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$;



$\lambda_1 = \frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}, \ \ \mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix},$

$\lambda_2 = \frac{1}{2} - \mathrm{i}\,\frac{\sqrt{3}}{2}, \ \ \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} - \mathrm{i}\,\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix},$

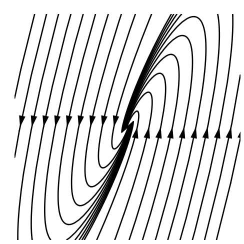
$u_1(t) = e^{-\,t/2}\left[\left(\frac{3}{2}\,c_1 - \frac{\sqrt{3}}{2}\,c_2\right)\cos\frac{\sqrt{3}}{2}\,t + \left(\frac{\sqrt{3}}{2}\,c_1 + \frac{3}{2}\,c_2\right)\sin\frac{\sqrt{3}}{2}\,t\right],$

$u_2(t) = e^{-\,t/2}\left[c_1\cos\frac{\sqrt{3}}{2}\,t + c_2\sin\frac{\sqrt{3}}{2}\,t\right];$

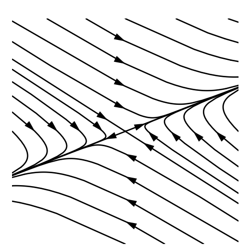
stable focus; asymptotically stable

10.3.2. (ii) $\mathbf{u}(t) = c_1\,e^{-\,t}\begin{pmatrix} 2\cos t - \sin t \\ 5\cos t \end{pmatrix} + c_2\,e^{-\,t}\begin{pmatrix} 2\sin t + \cos t \\ 5\sin t \end{pmatrix};$

stable focus; asymptotically stable.



10.3.3. (a) For the matrix $A = \begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix}$,



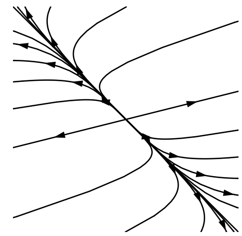
$\operatorname{tr} A = -3 < 0, \ \ \det A = -2 < 0, \ \ \Delta = 17 > 0,$

so this is an unstable saddle point.

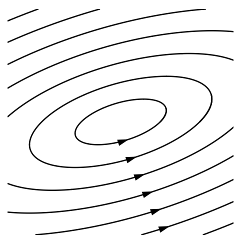
(c) For the matrix $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$,

$\operatorname{tr} A = 7 > 0$, $\det A = 6 > 0$, $\Delta = 25 > 0$,

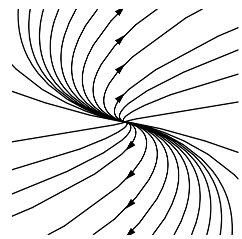
so this is an unstable node.



10.3.5. (b)



(d)



10.4.1. (a) $\begin{pmatrix} \frac{4}{3}e^{t} - \frac{1}{3}e^{-2t} & -\frac{1}{3}e^{t} + \frac{1}{3}e^{-2t} \\ \frac{4}{3}e^{t} - \frac{4}{3}e^{-2t} & -\frac{1}{3}e^{t} + \frac{4}{3}e^{-2t} \end{pmatrix}$, (c) $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$,

(e) $\begin{pmatrix} e^{2t}\cos t - 3e^{2t}\sin t & 2e^{2t}\sin t \\ -5e^{2t}\sin t & e^{2t}\cos t + 3e^{2t}\sin t \end{pmatrix}$.

10.4.2. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 2\sin t & \cos t & \sin t \\ 2\cos t - 2 & -\sin t & \cos t \end{pmatrix}$, (c) $\begin{pmatrix} e^{-2t} + t e^{-2t} & t e^{-2t} & t e^{-2t} \\ -1 + e^{-2t} & e^{-2t} & -1 + e^{-2t} \\ 1 - e^{-2t} - t e^{-2t} & -t e^{-2t} & 1 - t e^{-2t} \end{pmatrix}$.

10.4.3. Exercise 10.4.1: (a) $\det e^{tA} = e^{-t} = e^{t \operatorname{tr} A}$, (c) $\det e^{tA} = 1 = e^{t \operatorname{tr} A}$, (e) $\det e^{tA} = e^{4t} = e^{t \operatorname{tr} A}$. Exercise 10.4.2: (a) $\det e^{tA} = 1 = e^{t \operatorname{tr} A}$, (c) $\det e^{tA} = e^{-4t} = e^{t \operatorname{tr} A}$.

10.4.4. (b) $\mathbf{u}(t) = \begin{pmatrix} 3e^{-t} - 2e^{-3t} & -3e^{-t} + 3e^{-3t} \\ 2e^{-t} - 2e^{-3t} & -2e^{-t} + 3e^{-3t} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -6e^{-t} + 5e^{-3t} \\ -4e^{-t} + 5e^{-3t} \end{pmatrix}$.

10.4.5. (b) $\begin{pmatrix} e\cos\sqrt{2} & -\sqrt{2}\, e\sin\sqrt{2} \\ \frac{1}{\sqrt{2}}\, e\sin\sqrt{2} & e\cos\sqrt{2} \end{pmatrix}$, (d) $\begin{pmatrix} e & 0 & 0 \\ 0 & e^{-2} & 0 \\ 0 & 0 & e^{-5} \end{pmatrix}$.

10.4.8. There are none, since e^{tA} is always invertible.

10.4.11. The origin is an asymptotically stable if and only if all solutions tend to zero as $t \to \infty$. Thus, all columns of e^{tA} tend to $\mathbf{0}$ as $t \to \infty$, and hence $\lim_{t\to\infty} e^{tA} = \mathrm{O}$. Conversely, if $\lim_{t\to\infty} e^{tA} = \mathrm{O}$, then any solution has the form $\mathbf{u}(t) = e^{tA}\mathbf{c}$, and hence $\mathbf{u}(t) \to \mathbf{0}$ as $t \to \infty$, proving asymptotic stability.

10.4.13. (a) False, unless $A^{-1} = -A$.

10.4.15. Set $U(t) = A\,e^{tA}$, $V(t) = e^{tA}A$. Then, by the matrix Leibniz formula (10.41),

$$\dot U = A^2 e^{tA} = A\,U, \qquad \dot V = A\,e^{tA}A = A\,V,$$

while $U(0) = A = V(0)$. Thus $U(t)$ and $V(t)$ solve the same initial value problem, hence, by uniqueness, $U(t) = V(t)$ for all t. Alternatively, one can use the power series formula (10.47):

$$A\,e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,A^{n+1} = e^{tA}A.$$

10.4.20. Lemma 10.28 implies $\det e^{tA} = e^{t\,\mathrm{tr}\,A} = 1$ for all t if and only if $\mathrm{tr}\,A = 0$. (Even if $\mathrm{tr}\,A$ is allowed to be complex, by continuity the only way this could hold for all t is if $\mathrm{tr}\,A = 0$.)

♢ 10.4.25. (a) $\dfrac{d}{dt}\,\mathrm{diag}\,(e^{t d_1},\ldots e^{t d_n}) = \mathrm{diag}\,(d_1 e^{t d_1},\ldots,d_n e^{t d_n}) = D\,\mathrm{diag}\,(e^{t d_1},\ldots,e^{t d_n})$.
Moreover, at $t = 0$, we have $\mathrm{diag}\,(e^{0 d_1},\ldots,e^{0 d_n}) = \mathrm{I}$. Therefore, $\mathrm{diag}\,(e^{t d_1},\ldots,e^{t d_n})$ satisfies the defining properties of e^{tD}.

(c) 10.4.1: (a) $\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac43 e^t - \frac13 e^{-2t} & -\frac13 e^t + \frac13 e^{-2t} \\ \frac43 e^t - \frac43 e^{-2t} & -\frac13 e^t + \frac43 e^{-2t} \end{pmatrix}$;

(c) $\begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{\mathrm{i}t} & 0 \\ 0 & e^{-\mathrm{i}t} \end{pmatrix}\begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$;

(e) $\begin{pmatrix} \frac35 - \frac15\mathrm{i} & \frac35 + \frac15\mathrm{i} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{(2+\mathrm{i})t} & 0 \\ 0 & e^{(2-\mathrm{i})t} \end{pmatrix}\begin{pmatrix} \frac35 - \frac15\mathrm{i} & \frac35 + \frac15\mathrm{i} \\ 1 & 1 \end{pmatrix}^{-1}$
$$= \begin{pmatrix} e^{2t}\cos t - 3e^{2t}\sin t & 2e^{2t}\sin t \\ -5e^{2t}\sin t & e^{2t}\cos t + 3e^{2t}\sin t \end{pmatrix}.$$

10.4.2: (a)
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\mathrm{i} & \mathrm{i} \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\mathrm{i}t} & 0 \\ 0 & 0 & e^{-\mathrm{i}t} \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\mathrm{i} & \mathrm{i} \\ 2 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2\sin t & \cos t & \sin t \\ 2\cos t - 2 & -\sin t & \cos t \end{pmatrix};$$

(c) not diagonalizable.

♢ 10.4.28. (a) If $U(t) = C\,e^{tB}$, then $\dfrac{dU}{dt} = C\,e^{tB}B = U\,B$, and so U satisfies the differential equation. Moreover, $C = U(0)$. Thus, $U(t)$ is the unique solution to the initial value problem $\dot U = U\,B$, $U(0) = C$, where the initial value C is arbitrary.

10.4.32. (b) $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ — shear transformations in the y direction. The trajectories are lines parallel to the y-axis. Points on the y-axis are fixed.

(d) $\begin{pmatrix} \cos 2t & -\sin 2t \\ 2\sin 2t & 2\cos 2t \end{pmatrix}$ — elliptical rotations around the origin. The trajectories are the ellipses $x^2 + \frac14 y^2 = c$. The origin is fixed.

10.4.33. (b) $\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ — shear transformations in the x direction, with magnitude proportional to the z coordinate. The trajectories are lines parallel to the x axis. Points on the xy plane are fixed.

(d) $\begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^t \end{pmatrix}$ — spiral motions around the z axis. The trajectories are the positive and negative z axes, circles in the xy plane, and cylindrical spirals (helices) winding around the z axis while going away from the xy pane at an exponentially increasing rate. The only fixed point is the origin.

10.4.35.

$$\left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}, \qquad \left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix},$$

$$\left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & -2 \\ -8 & 0 \end{pmatrix}, \qquad \left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},$$

10.4.41. In the matrix system $\dfrac{dU}{dt} = AU$, the equations in the last row are $\dfrac{du_{nj}}{dt} = 0$ for $j = 1, \ldots, n$, and hence the last row of $U(t)$ is constant. In particular, for the exponential matrix solution $U(t) = e^{tA}$ the last row must equal the last row of the identity matrix $U(0) = \mathrm{I}$, which is $\mathbf{e}_n^T$.

♢ 10.4.43. (a) $\begin{pmatrix} x+t \\ y \end{pmatrix}$: translations in x direction.

(c) $\begin{pmatrix} (x+1)\cos t - y \sin t - 1 \\ (x+1)\sin t + y\cos t \end{pmatrix}$: rotations around the point $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

10.4.44. (a) If $U \neq \{\mathbf{0}\}$, then the system has eigenvalues with positive real part corresponding to exponentially growing solutions and so the origin is unstable. If $C \neq \{\mathbf{0}\}$, then the system has eigenvalues with zero real part corresponding to either bounded solutions, which are stable but not asymptotically stable modes, or, if the eigenvalue is incomplete, polynomial growing unstable modes.

10.4.45. (a) $S = U = \varnothing$, $C = \mathbb{R}^2$;

(b) $S = \operatorname{span}\begin{pmatrix} \frac{2-\sqrt{7}}{3} \\ 1 \end{pmatrix}$, $U = \operatorname{span}\begin{pmatrix} \frac{2+\sqrt{7}}{3} \\ 1 \end{pmatrix}$, $C = \varnothing$;

(d) $S = \operatorname{span}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$, $U = \operatorname{span}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $C = \operatorname{span}\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$.

10.4.47. (b) $u_1(t) = e^{t-1} - e^t + t\,e^t$, $u_2(t) = e^{t-1} - e^t + t\,e^t$;

(d) $u(t) = \frac{13}{16}e^{4t} + \frac{3}{16} - \frac{1}{4}t$, $v(t) = \frac{13}{16}e^{4t} - \frac{29}{16} + \frac{3}{4}t$.

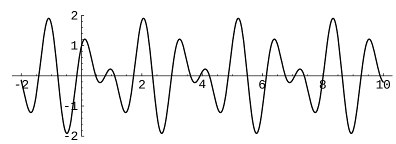
10.4.48. (a) $u_1(t) = \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{2} - \frac{1}{2}t,$

$$u_2(t) = 2e^{-t} - \tfrac{1}{2}\cos 2t - \tfrac{1}{4}\sin 2t - \tfrac{3}{2} + \tfrac{3}{2}t,$$

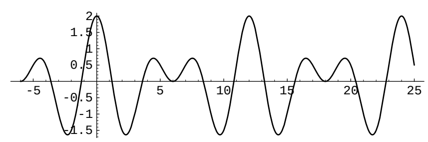
$$u_3(t) = 2e^{-t} - \tfrac{1}{4}\cos 2t - \tfrac{3}{4}\sin 2t - \tfrac{7}{4} + \tfrac{3}{2}t.$$

10.5.1. The vibrational frequency is $\omega = \sqrt{21/6} \simeq 1.87083$, and so the number of hertz is $\omega/(2\pi) \simeq .297752$.

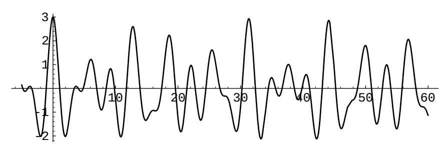
10.5.3. (a) Periodic of period π:



(c) Periodic of period 12:



(f) Quasi-periodic:



10.5.5. (a) $\sqrt{2}\,,\sqrt{7}\,$; (b) 4 — each eigenvalue gives two linearly independent solutions;

(c) $\mathbf{u}(t) = r_1\cos(\sqrt{2}\,t - \delta_1)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + r_2\cos(\sqrt{7}\,t - \delta_2)\begin{pmatrix} -1 \\ 2 \end{pmatrix}$;

(d) The solution is periodic if only one frequency is excited, i.e., $r_1 = 0$ or $r_2 = 0$; all other solutions are quasiperiodic.

10.5.7. (a) $u(t) = r_1\cos(t - \delta_1) + r_2\cos(\sqrt{5}\,t - \delta_2)$, $v(t) = r_1\cos(t - \delta_1) - r_2\cos(\sqrt{5}\,t - \delta_2)$;

(c) $\mathbf{u}(t) = (\,r_1\cos(t - \delta_1), r_2\cos(2t - \delta_2), r_3\cos(3t - \delta_1)\,)^T$.

♠ 10.5.11. (a) The vibrational frequencies and eigenvectors are

$$\omega_1 = \sqrt{2 - \sqrt{2}} = .7654, \qquad \omega_2 = \sqrt{2} = 1.4142,, \qquad \omega_3 = \sqrt{2 + \sqrt{2}} = 1.8478,$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

Thus, in the slowest mode, all three masses are moving in the same direction, with the middle mass moving $\sqrt{2}$ times farther; in the middle mode, the two outer masses are moving in opposing directions by equal amounts, while the middle mass remains still; in the fastest mode, the two outer masses are moving in tandem, while the middle mass is moving farther in an opposing direction.

10.5.16. (a) $u(t) = a\,t + b + 2\,r\cos(\sqrt{5}\,t - \delta), \quad v(t) = -2\,a\,t - 2\,b + r\cos(\sqrt{5}\,t - \delta).$
The unstable mode consists of the terms with a in them; it will not be excited if the initial conditions satisfy $\dot u(t_0) - 2\,\dot v(t_0) = 0$.

$$(c)\quad u(t) = -2\,a\,t - 2\,b - \tfrac{1-\sqrt{13}}{4}\, r_1 \cos\left(\sqrt{\tfrac{7+\sqrt{13}}{2}}\, t - \delta_1\right) - \tfrac{1+\sqrt{13}}{4}\, r_2 \cos\left(\sqrt{\tfrac{7-\sqrt{13}}{2}}\, t - \delta_2\right),$$

$$v(t) = -2\,a\,t - 2\,b + \tfrac{3-\sqrt{13}}{4}\, r_1 \cos\left(\sqrt{\tfrac{7+\sqrt{13}}{2}}\, t - \delta_1\right) + \tfrac{3+\sqrt{13}}{4}\, r_2 \cos\left(\sqrt{\tfrac{7-\sqrt{13}}{2}}\, t - \delta_2\right),$$

$$w(t) = a\,t + b + r_1 \cos\left(\sqrt{\tfrac{7+\sqrt{13}}{2}}\, t - \delta_1\right) + r_2 \cos\left(\sqrt{\tfrac{7-\sqrt{13}}{2}}\, t - \delta_2\right).$$

The unstable mode is the term containing a; it will not be excited if the initial conditions satisfy $-2\,\dot u(t_0) - 2\,\dot v(t_0) + \dot w(t_0) = 0$.

10.5.17. (a) $Q = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$

(b) Yes, because K is symmetric and has all positive eigenvalues.

(c) $\mathbf{u}(t) = \left(\cos\sqrt{2}\,t, \dfrac{1}{\sqrt{2}}\,\sin\sqrt{2}\,t, \cos\sqrt{2}\,t \right)^T.$

(d) The solution $\mathbf{u}(t)$ is periodic with period $\sqrt{2}\,\pi$.

(e) No — since the frequencies $2, \sqrt{2}$ are not rational multiples of each other, the general solution is quasi-periodic.

♠ 10.5.22. (a) Frequencies: $\omega_1 = \sqrt{\frac{3}{2} - \frac{1}{2}\sqrt{5}} = .61803$, $\omega_2 = 1$, $\omega_3 = \sqrt{\frac{3}{2} + \frac{1}{2}\sqrt{5}} = 1.618034$;
stable eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} 2-\sqrt{5} \\ -1 \\ -2+\sqrt{5} \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2+\sqrt{5} \\ 1 \\ -2-\sqrt{5} \\ 1 \end{pmatrix}$; unstable eigenvector: $\mathbf{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$. In the lowest frequency mode, the nodes vibrate up and towards each other and then down and away, the horizontal motion being less pronounced than the vertical; in the next mode, the nodes vibrate in the directions of the diagonal bars, with one moving towards the support while the other moves away; in the highest frequency mode, they vibrate up and away from each other and then down and towards, with the horizontal motion significantly more than the vertical; in the unstable mode the left node moves down and to the right, while the right-hand node moves at the same rate up and to the right.

10.5.27. (a) $\mathbf{u}(t) = r_1 \cos\left(\tfrac{1}{\sqrt{2}}\, t - \delta_1\right) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + r_2 \cos\left(\sqrt{\tfrac{5}{3}}\, t - \delta_2\right) \begin{pmatrix} -3 \\ 1 \end{pmatrix},$

(c) $\mathbf{u}(t) = r_1 \cos\left(\sqrt{\tfrac{3-\sqrt{3}}{2}}\, t - \delta_1\right) \begin{pmatrix} \frac{1+\sqrt{3}}{2} \\ 1 \end{pmatrix} + r_2 \cos\left(\sqrt{\tfrac{3+\sqrt{3}}{2}}\, t - \delta_2\right) \begin{pmatrix} \frac{1-\sqrt{3}}{2} \\ 1 \end{pmatrix}.$

10.5.28. $u_1(t) = \frac{\sqrt{3}-1}{2\sqrt{3}} \cos\sqrt{\frac{3-\sqrt{3}}{2}}\, t + \frac{\sqrt{3}+1}{2\sqrt{3}} \cos\sqrt{\frac{3+\sqrt{3}}{2}}\, t,$

$$u_2(t) = \tfrac{1}{2\sqrt{3}} \cos\sqrt{\tfrac{3-\sqrt{3}}{2}}\, t - \tfrac{1}{2\sqrt{3}} \cos\sqrt{\tfrac{3+\sqrt{3}}{2}}\, t.$$

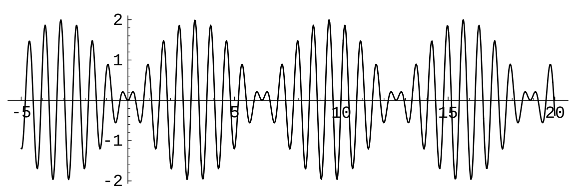
♣ 10.5.30. (a) We place the oxygen molecule at the origin, one hydrogen at $(1,0)^T$ and the other at $(\cos\theta, \sin\theta)^T = (-0.2588, 0.9659)^T$ with $\theta = \frac{105}{180}\pi = 1.8326$ radians. There are two independent vibrational modes, whose fundamental frequencies are $\omega_1 = 1.0386$, $\omega_2 = 1.0229$, with corresponding eigenvectors $\mathbf{v}_1 = (.0555, -.0426, -.7054, 0., -.1826, .6813)^T$, $\mathbf{v}_2 = (-.0327, -.0426, .7061, 0., -.1827, .6820)^T$. Thus, the (very slightly) higher frequency mode has one hydrogen atoms moving towards and the other away from the oxygen, which also slightly vibrates, and then all reversing their motion, while in the lower frequency mode, they simultaneously move towards and then away from the oxygen atom.

10.5.37. The solution is $u(t) = \frac{1}{4}(v+5)e^{-t} - \frac{1}{4}(v+1)e^{-5t}$, where $v = \dot u(0)$ is the initial velocity. This vanishes when $e^{4t} = \dfrac{v+1}{v+5}$, which happens when $t = t_\star > 0$ provided $\dfrac{v+1}{v+5} > 1$, and so the initial velocity must satisfy $v < -5$.

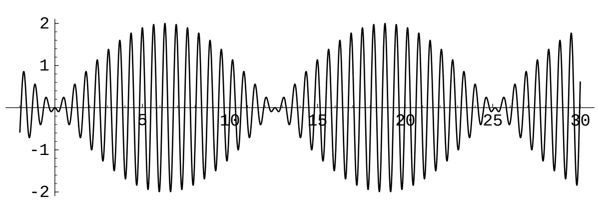
10.5.38. (a) $u(t) = t\,e^{-3t}$; critically damped. (c) $u(t) = \frac{1}{4}\sin 4(t-1)$; undamped.
(e) $u(t) = 4e^{-t/2} - 2e^{-t}$; overdamped.

10.5.39. (a) By Hooke's Law, the spring stiffness is $k = 16/6.4 = 2.5$. The mass is $16/32 = .5$. The equations of motion are $.5\,\ddot u + 2.5\,u = 0$. The natural frequency is $\omega = \sqrt{5} = 2.23607$. (b) The solution to the initial value problem $.5\,\ddot u + \dot u + 2.5\,u = 0$, $u(0) = 2$, $\dot u(0) = 0$, is $u(t) = e^{-t}(2\cos 2t + \sin 2t)$. (c) The system is underdamped, and the vibrations are less rapid than the undamped system.

10.6.1. (a) $\cos 8t - \cos 9t = 2\sin\frac{1}{2}t\ \sin\frac{17}{2}t$; fast frequency: $\frac{17}{2}$, beat frequency: $\frac{1}{2}$.



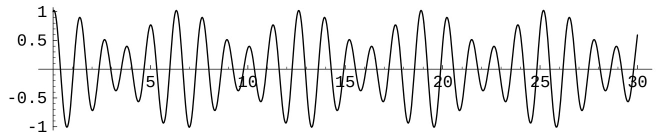
(c) $\cos 10t + \cos 9.5t = 2\sin .25t\ \sin 9.75t$; fast frequency: 9.75, beat frequency: .25.



10.6.2. (a) $u(t) = \frac{1}{27}\cos 3t - \frac{1}{27}\cos 6t$, (c) $u(t) = \frac{1}{2}\sin 2t + e^{-t/2}\left(\cos\frac{\sqrt{15}}{2}t - \frac{\sqrt{15}}{5}\sin\frac{\sqrt{15}}{2}t\right)$.

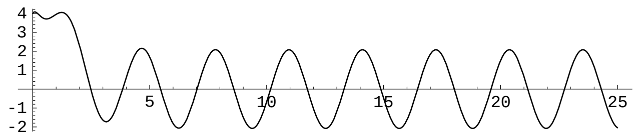
10.6.3. (a) $u(t) = \frac{1}{3}\cos 4t + \frac{2}{3}\cos 5t + \frac{1}{5}\sin 5t$;

undamped periodic motion with fast frequency 4.5 and beat frequency .5:



(c) $u(t) = -\frac{60}{29}\cos 2t + \frac{5}{29}\sin 2t - \frac{56}{29}e^{-5t} + 8e^{-t}$;

the transient is an overdamped motion; the persistent motion is periodic:



10.6.4. In general, by (10.107), the maximal allowable amplitude is

$$\alpha = \sqrt{m^2(\omega^2-\gamma^2)^2 + \beta^2\gamma^2} = \sqrt{625\,\gamma^4 - 49.9999\,\gamma^2 + 1},$$

which, in the particular cases is (a) .0975, (b) .002.

10.6.6. The solution to $.5\,\ddot{u} + \dot{u} + 2.5\,u = 2\cos 2t$, $u(0) = 2$, $\dot{u}(0) = 0$, is

$$\begin{aligned} u(t) &= \tfrac{4}{17}\cos 2t + \tfrac{16}{17}\sin 2t + e^{-t}\left(\tfrac{30}{17}\cos 2t - \tfrac{1}{17}\sin 2t\right) \\ &= .9701\cos(2t - 1.3258) + 1.7657\,e^{-t}\cos(2t + .0333). \end{aligned}$$

The solution consists of a persistent periodic vibration at the forcing frequency of 2, with a phase lag of $\tan^{-1} 4 = 1.32582$ and amplitude $4/\sqrt{17} = .97014$, combined with a transient vibration at the same frequency with exponentially decreasing amplitude.

10.6.10. (b) Overdamped, (c) critically damped, (d) underdamped.

10.6.11. (b) $u(t) = \frac{3}{2}e^{-t/3} - \frac{1}{2}e^{-t}$, (d) $u(t) = e^{-t/5}\cos\frac{1}{10}t + 2e^{-t/5}\sin\frac{1}{10}t$.

10.6.12.
$$\begin{aligned} u(t) &= \tfrac{165}{41}e^{-t/4}\cos\tfrac{1}{4}t - \tfrac{91}{41}e^{-t/4}\sin\tfrac{1}{4}t - \tfrac{124}{41}\cos 2t + \tfrac{32}{41}\sin 2t \\ &= 4.0244\,e^{-.25t}\cos .25t - 2.2195\,e^{-.25t}\sin .25t - 3.0244\cos 2t + .7805\sin 2t. \end{aligned}$$

10.6.14. (a) .02, (b) 2.8126.

10.6.17. $\mathbf{u}(t) =$

(b) $\sin 3t\begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} + r_1\cos(\sqrt{4+\sqrt{5}}\,t - \delta_1)\begin{pmatrix} -1-\sqrt{5} \\ 2 \end{pmatrix} + r_2\cos(4-\sqrt{5}\,t - \delta_2)\begin{pmatrix} -1+\sqrt{5} \\ 2 \end{pmatrix}$,

(d) $\cos\frac{1}{2}t\begin{pmatrix} \frac{2}{17} \\ -\frac{12}{17} \end{pmatrix} + r_1\cos(\sqrt{\tfrac{5}{3}}\,t - \delta_1)\begin{pmatrix} -3 \\ 1 \end{pmatrix} + r_2\cos(\tfrac{1}{\sqrt{2}}\,t - \delta_2)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

10.6.18. (a) The resonant frequencies are $\sqrt{\frac{3-\sqrt{3}}{2}} = .796225$, $\sqrt{\frac{3+\sqrt{3}}{2}} = 1.53819$.

(b) For example, a forcing function of the form $\cos\left(\sqrt{\frac{3+\sqrt{3}}{2}}\, t\right)\mathbf{w}$, where $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is not orthogonal to the eigenvector $\begin{pmatrix} -1-\sqrt{3} \\ 1 \end{pmatrix}$, so $w_2 \neq (1+\sqrt{3}\,)w_1$, will excite resonance.

♣ 10.6.20. In each case, you need to force the system by $\cos(\omega\, t)\mathbf{a}$ where $\omega^2 = \lambda$ is an eigenvalue and $\mathbf{a}$ is orthogonal to the corresponding eigenvector. In order not to excite an instability, $\mathbf{a}$ needs to also be orthogonal to the kernel of the stiffness matrix spanned by the unstable mode vectors.

(a) Resonant frequencies: $\omega_1 = .5412$, $\omega_2 = 1.1371$, $\omega_3 = 1.3066$, $\omega_4 = 1.6453$;

eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} .6533 \\ .2706 \\ .6533 \\ -.2706 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} .2706 \\ .6533 \\ -.2706 \\ .6533 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} .2706 \\ -.6533 \\ .2706 \\ .6533 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} -.6533 \\ .2706 \\ .6533 \\ .2706 \end{pmatrix}$;

no unstable modes.

(c) Resonant frequencies: $\omega_1 = .3542$, $\omega_2 = .9727$, $\omega_3 = 1.0279$, $\omega_4 = 1.6894$, $\omega_5 = 1.7372$;

eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} -.0989 \\ -.0706 \\ 0 \\ -.9851 \\ .0989 \\ -.0706 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -.1160 \\ .6780 \\ .2319 \\ 0 \\ -.1160 \\ -.6780 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} .1251 \\ -.6940 \\ 0 \\ .0744 \\ -.1251 \\ -.6940 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} .3914 \\ .2009 \\ -.7829 \\ 0 \\ .3914 \\ -.2009 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} .6889 \\ .1158 \\ 0 \\ -.1549 \\ -.6889 \\ .1158 \end{pmatrix};$$

unstable mode: $\mathbf{z} = (\,1, 0, 1, 0, 1, 0\,)^T$. To avoid exciting the unstable mode, the initial velocity must be orthogonal to the null eigenvector: $\mathbf{z} \cdot \dot{\mathbf{u}}(t_0) = 0$, i.e., there is no net horizontal velocity of the masses.

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