Class 10: Confidence intervals

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Frequentist vs. Bayesian

Sigh…

When you ask the same question,
you get the same answer

Common statistical questions

- [x] Am I confident this is a real signal?
- [ ] I am confident I saw something, what was the real signal strength/level?
- [ ] I didn’t see anything, how faint/small must the signal have been for me <u>not</u> to see it?

Uncertainty on a Measurement

Uncertainty on a measurement

- What if you have a high σ detection, and now you want to know what the accuracy of your measurement is?

Uncertainty on a measurement

Example statistical question:

- If I performed the same measurement many times, what range of signal values would I observe?

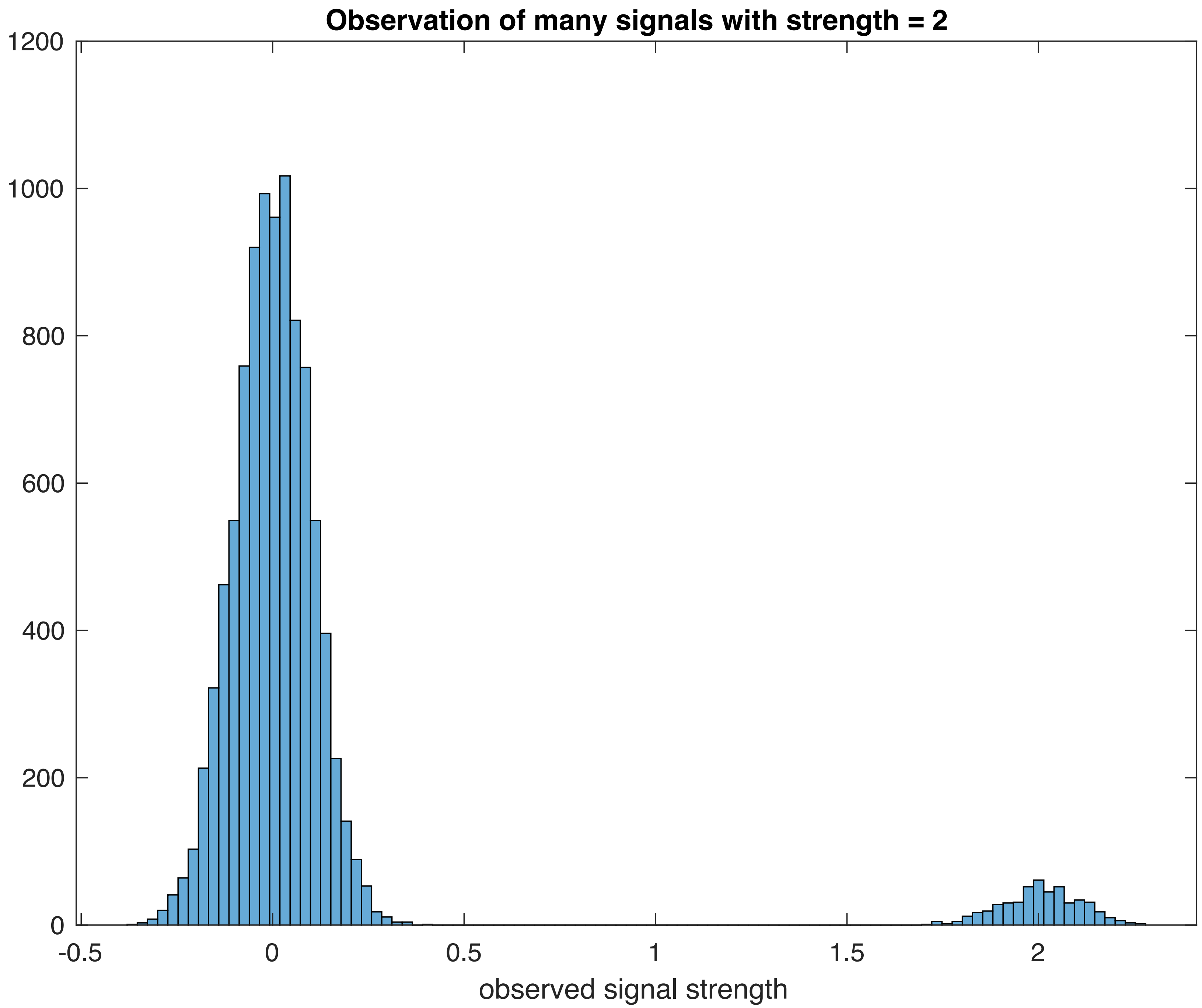
Uncertainty on a measurement

Simulation approach

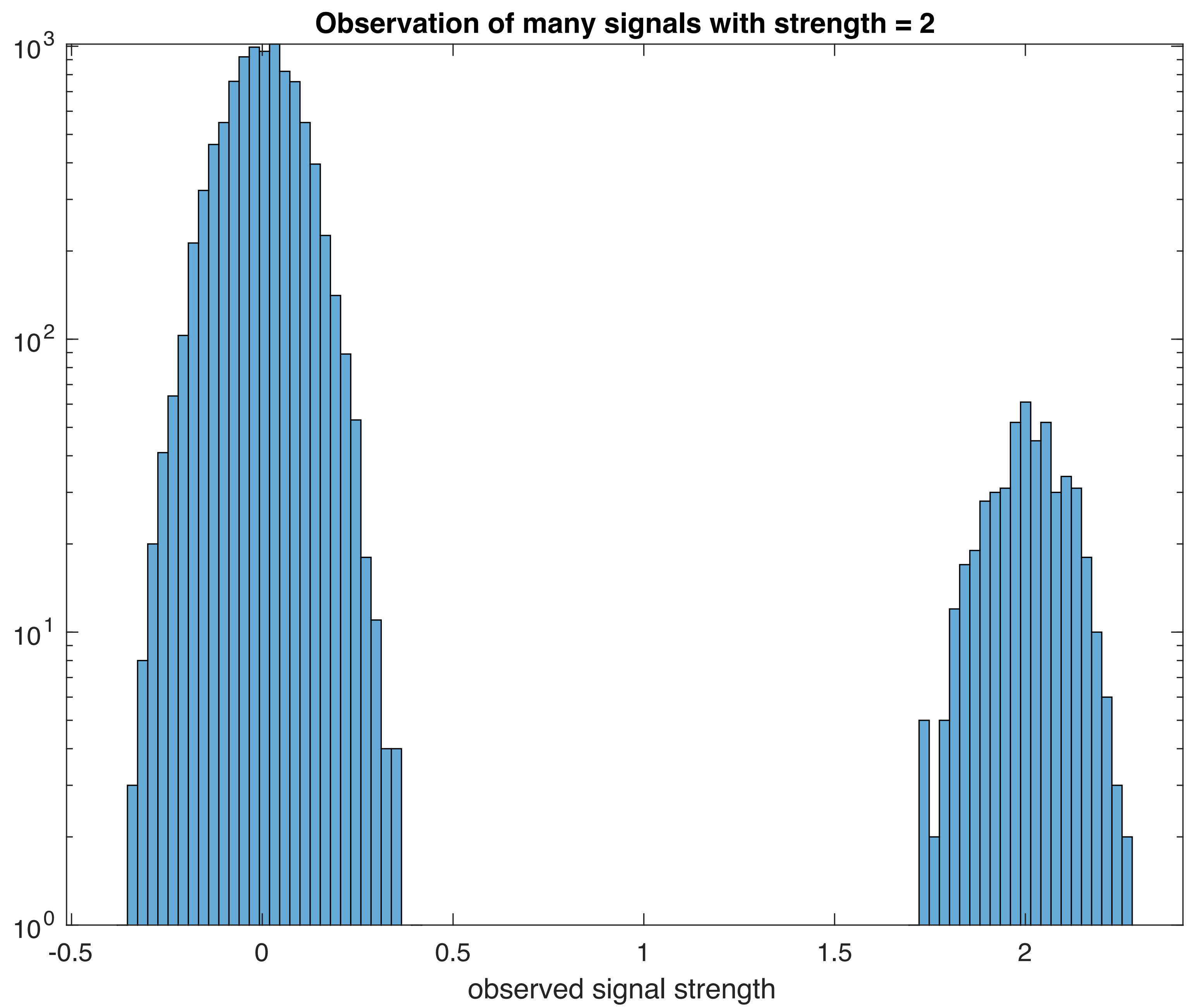
```
size = 10000;
noise = randn(1,size)*0.1;
signal = zeros(1,size);
signal(randi(numel(signal),[1,500])) = 2;

obssiganl = noise + signal;
```

Uncertainty on a measurement



Uncertainty on a measurement



Uncertainty on a measurement

Statistical question:

- If I performed the same measurement many times, what range of signal values would I observe?

Even if the signal strength is constant, we observe a range of measurements

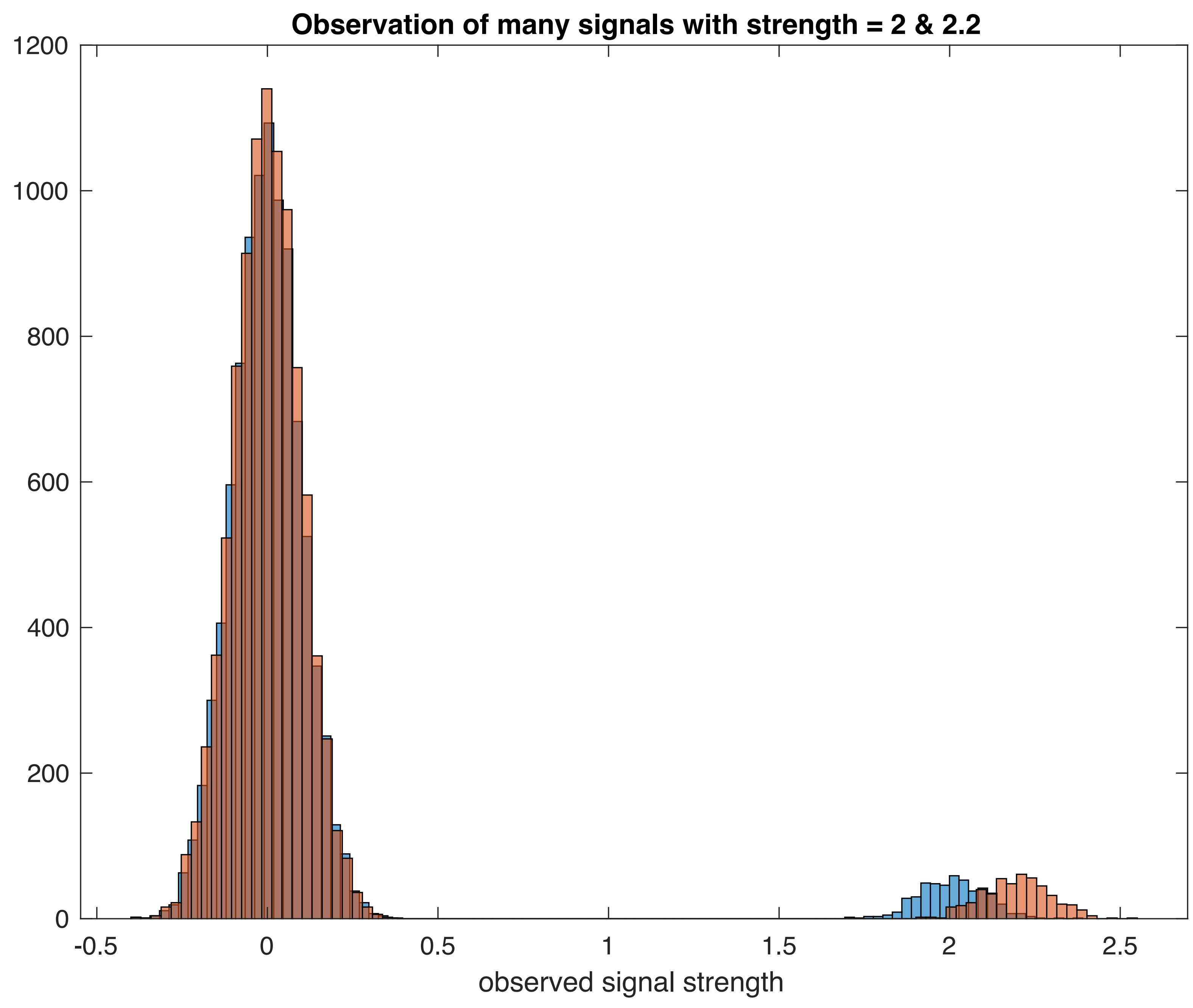
Uncertainty on a measurement

Turn statistical question around:

- If I measure a signal once, what range of true signal strengths could have given me the same observation?

Test with simulation of two input signals (2.0 & 2.2)

Uncertainty on a measurement



An asside in math notation

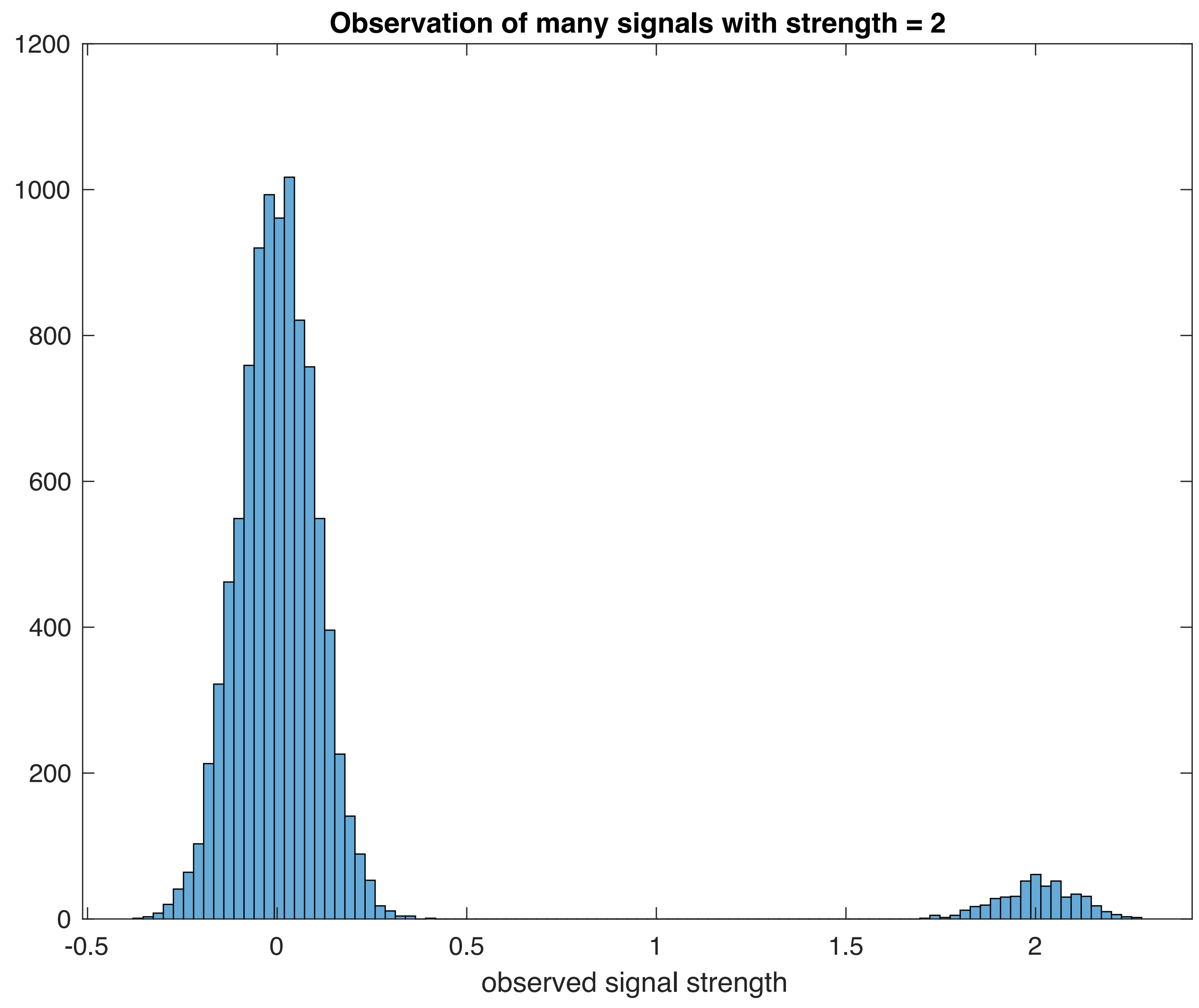
Read as:

- Given a particular true signal, what is the probability of getting a particular data value?

$$P(\text{data} \mid \text{signal}_{\text{T}})$$

An asside in math notation

$$P(\text{data} \mid \text{signal}_{\text{T}})$$



Two separate questions

A. If I performed the same measurement many times, what range of observed signal values (data) would I observe?
$P(\mathrm{data} \,|\, \mathrm{signal}_\mathrm{T})$

B. If I measure a signal once (data), what is the probability of the true signal strengths? $P(\mathrm{signal}_\mathrm{T} \,|\, \mathrm{data})$

Bayes' theorem

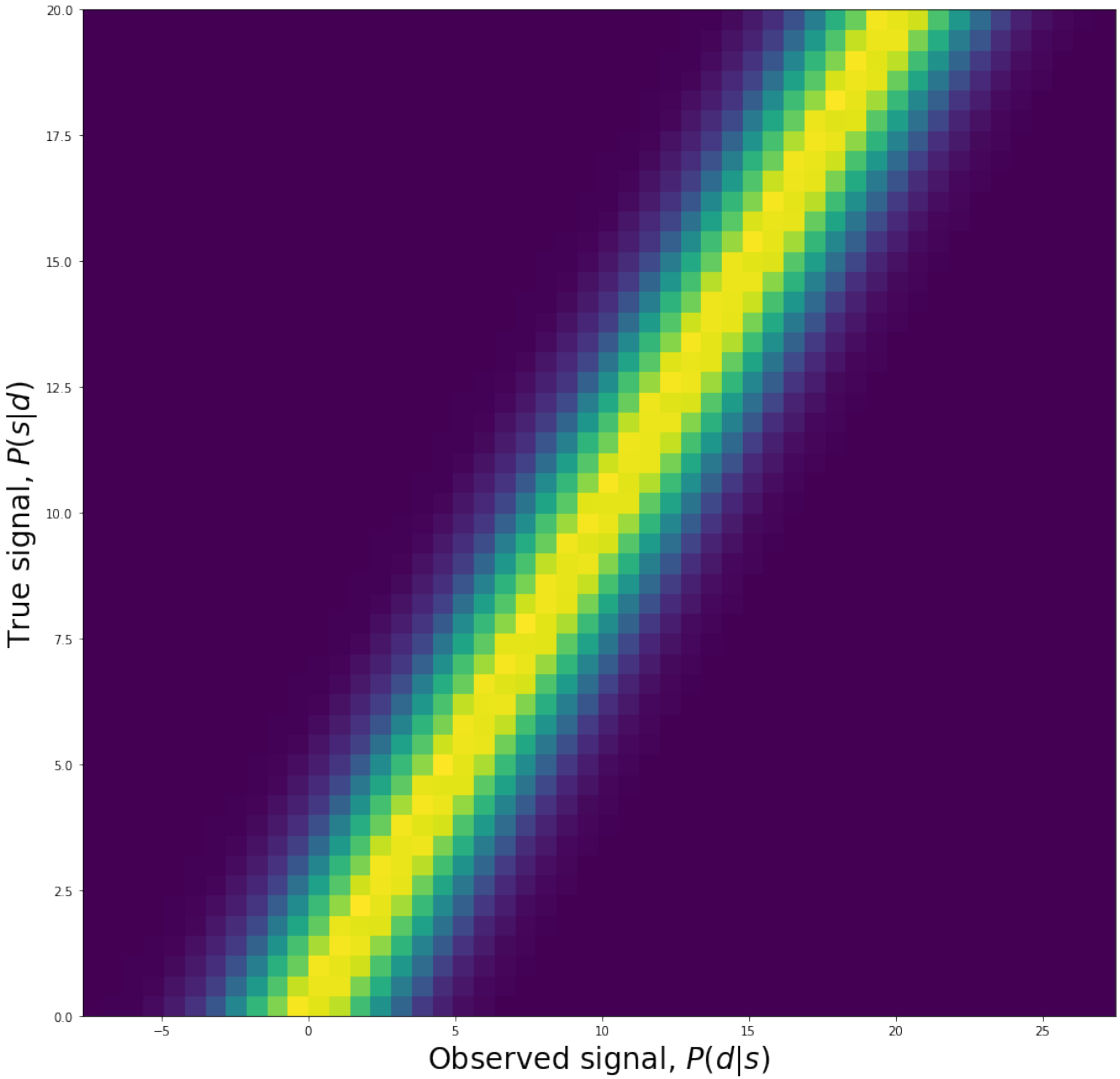
- Formally can change questions and calculate the desired $P(\mathrm{signal_T} \mid \mathrm{data})$

- In practice must be used with great care

$$P(s \mid d) = \frac{P(d \mid s)P(s)}{P(d)}$$

Confidence interval

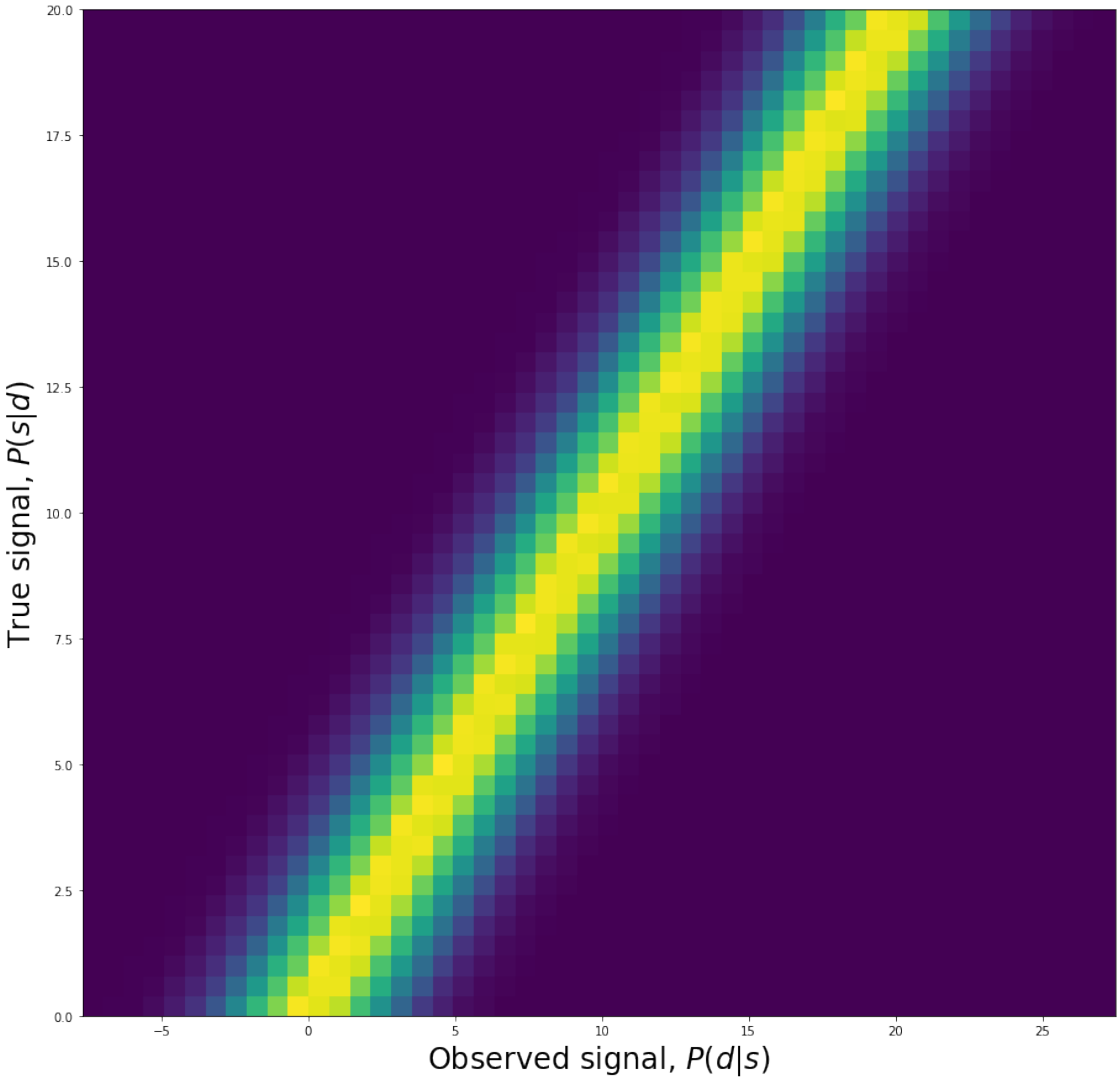
Simulated observations



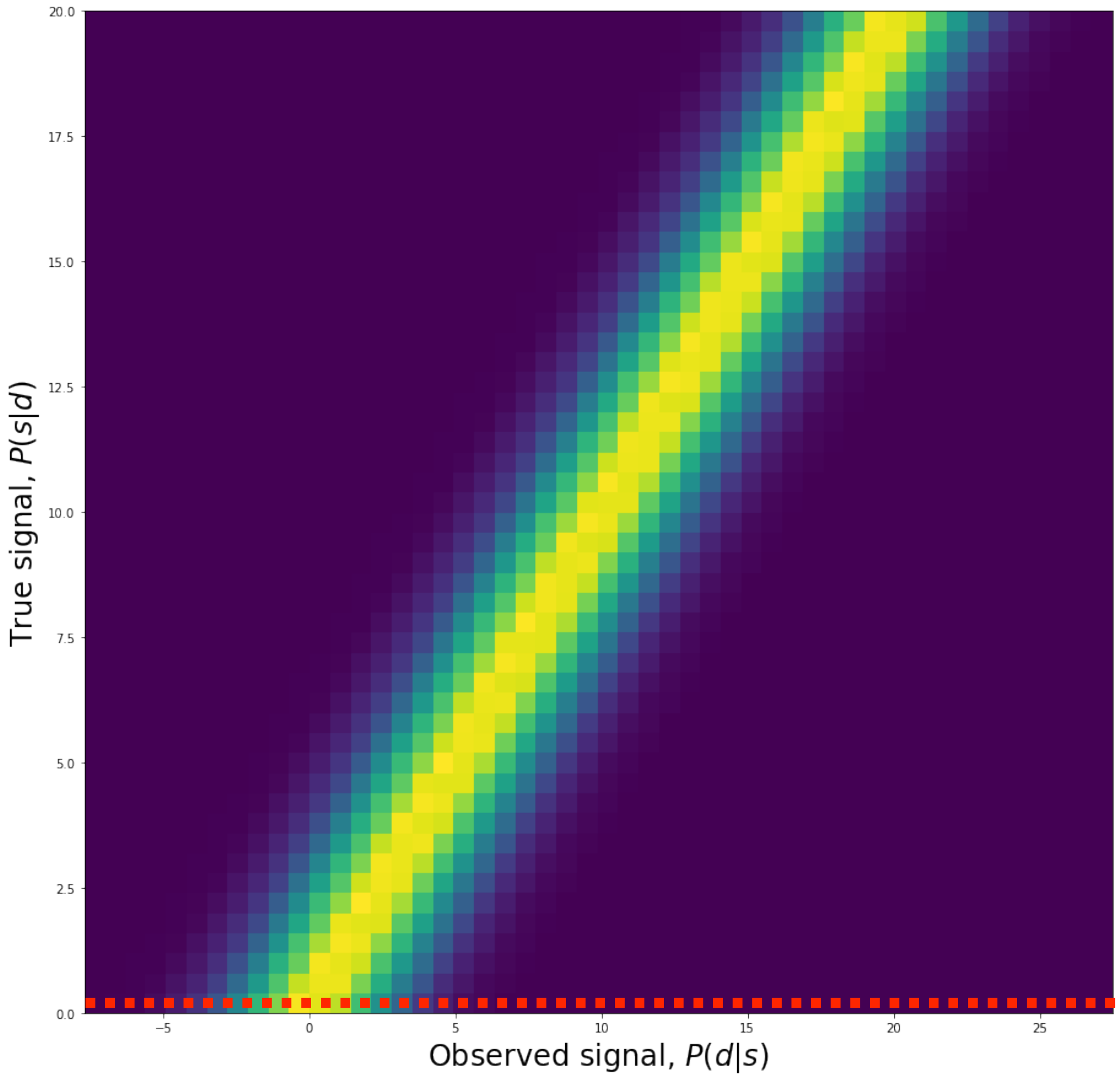
How to make

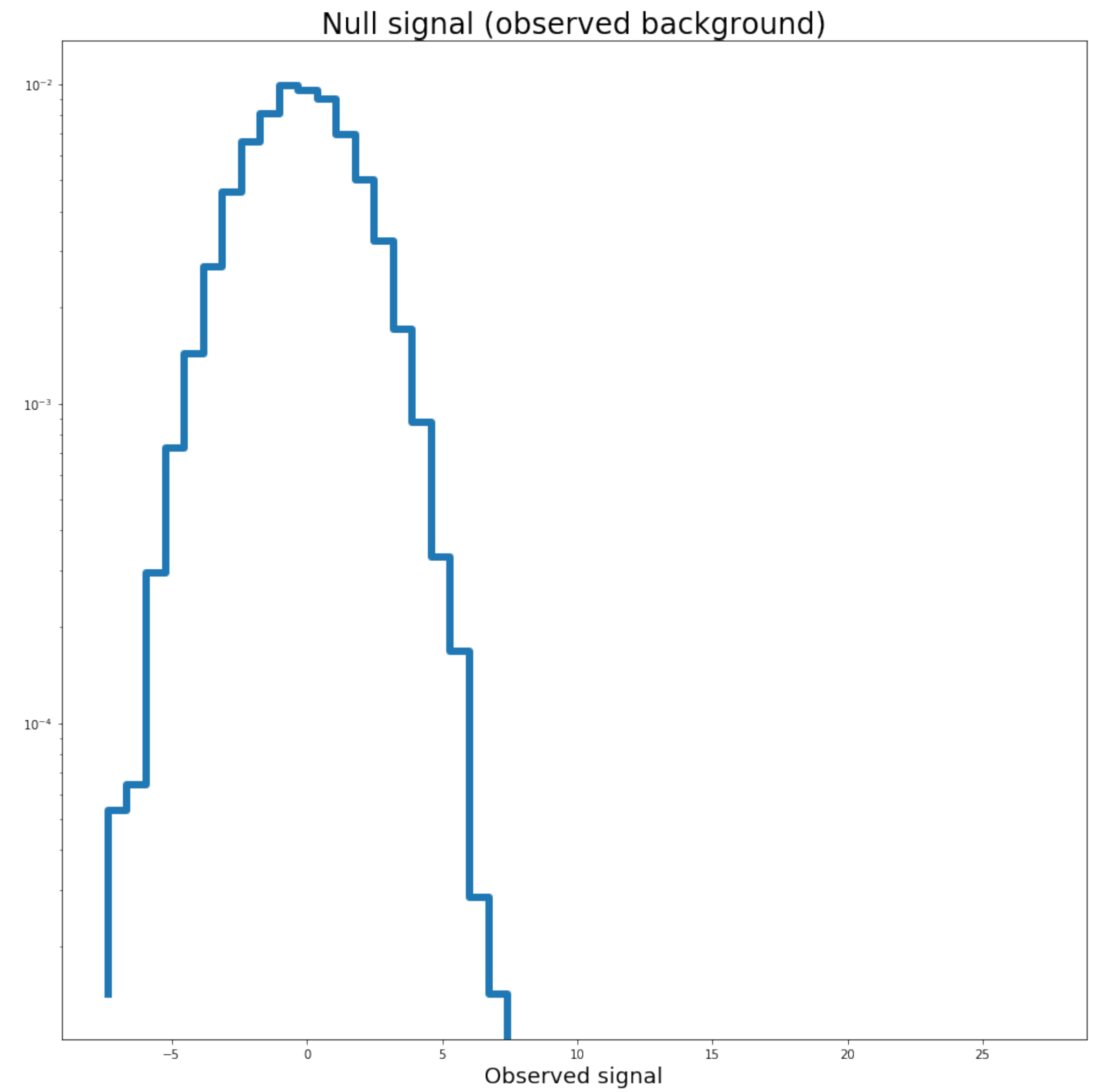
- Start with background (model or data)
- Inject fake signals of varying strength
- Measure observed signal
- Histogram true signal vs. observed signal

Simulated observations



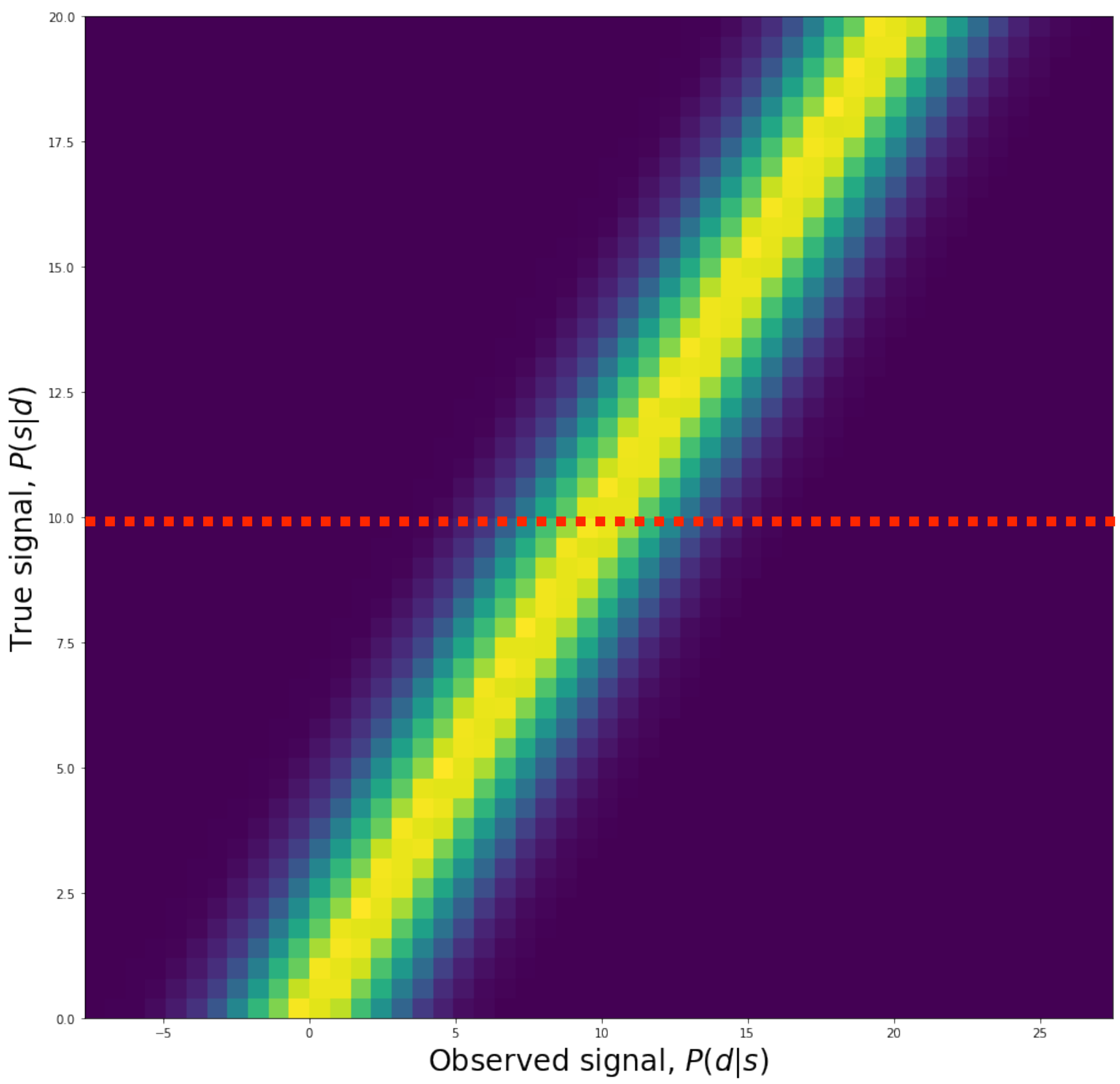
Slices



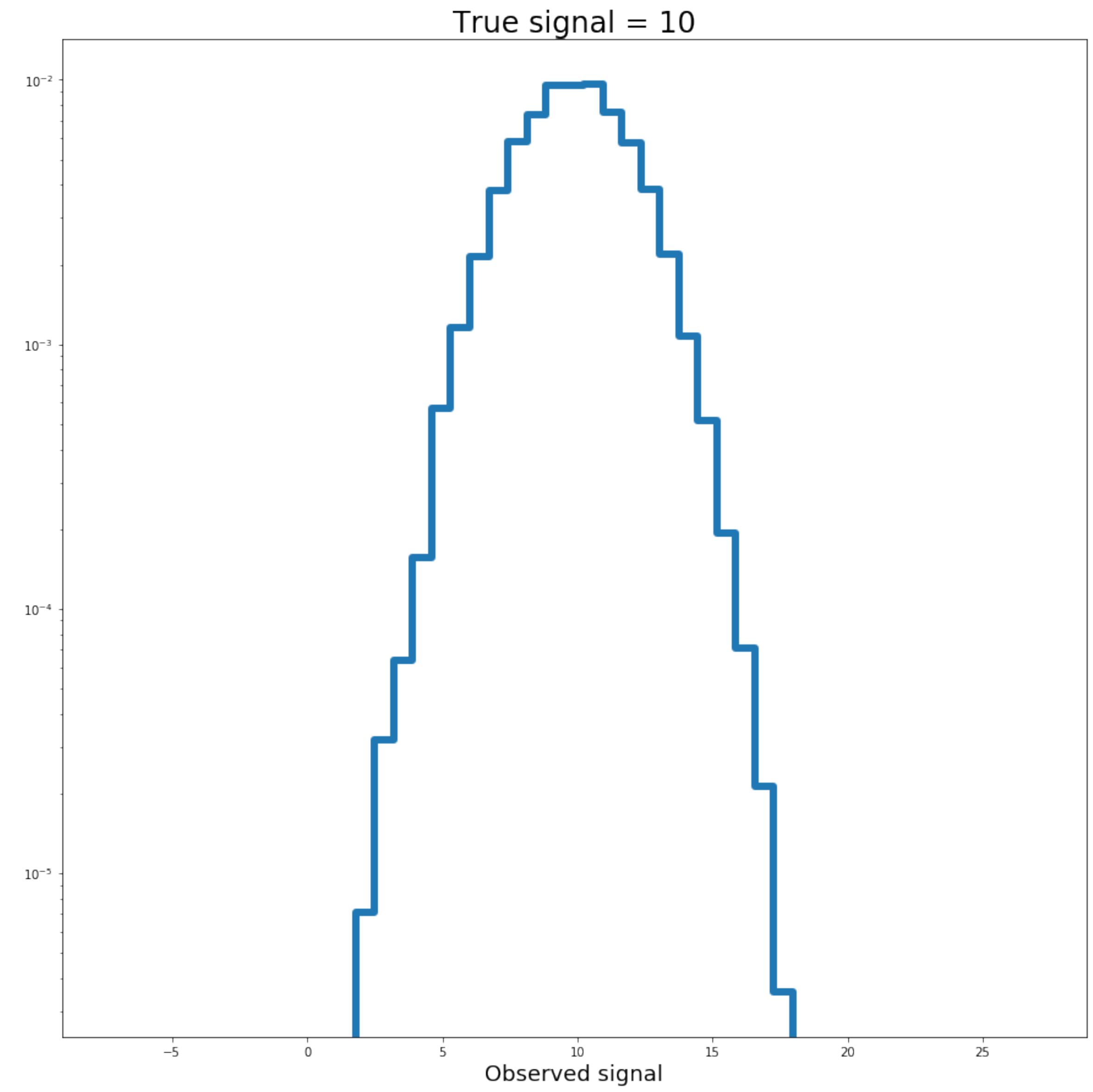


$$P(d \mid s)$$

Slices



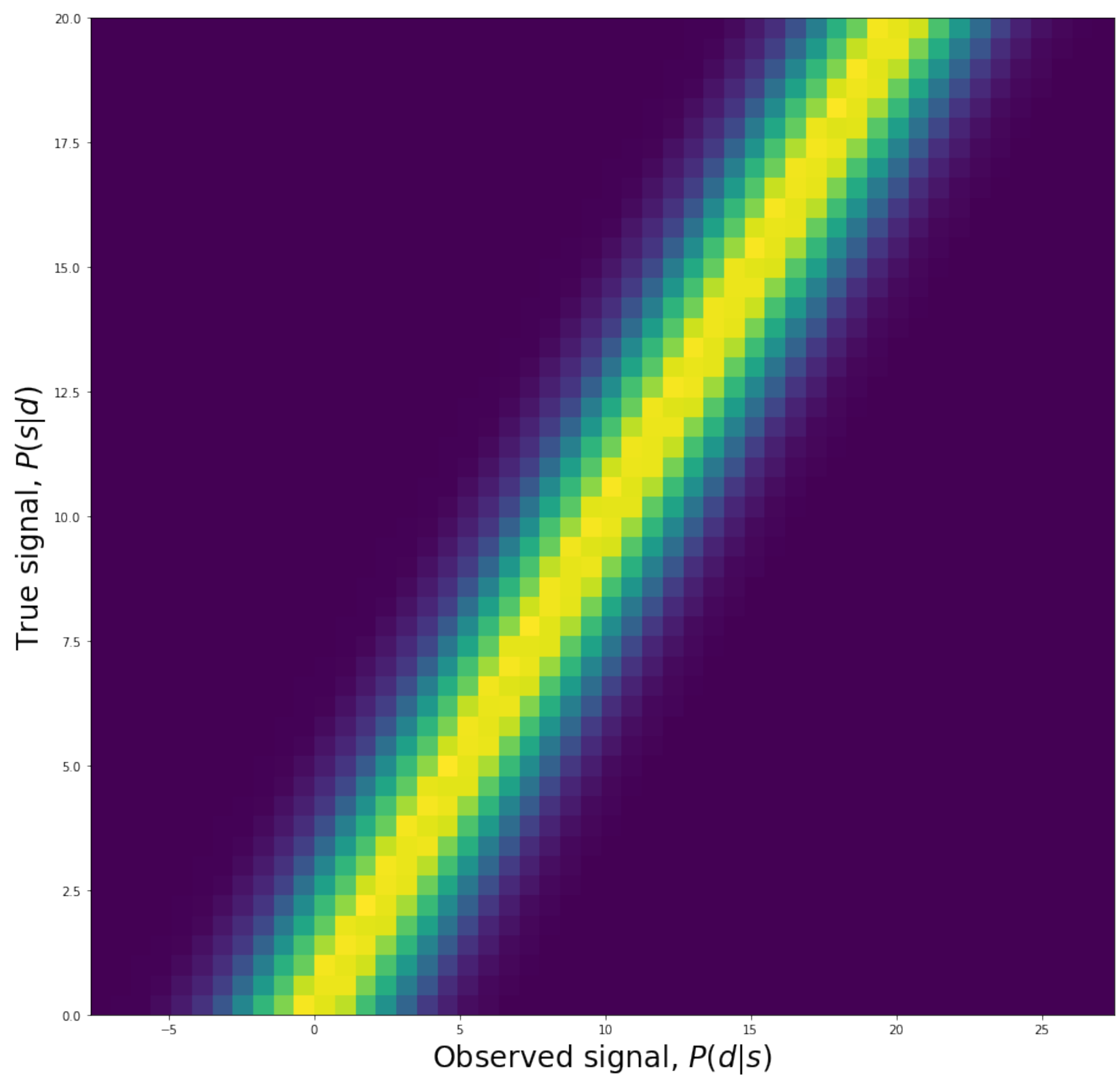
True signal, $P(s|d)$
Observed signal, $P(d|s)$



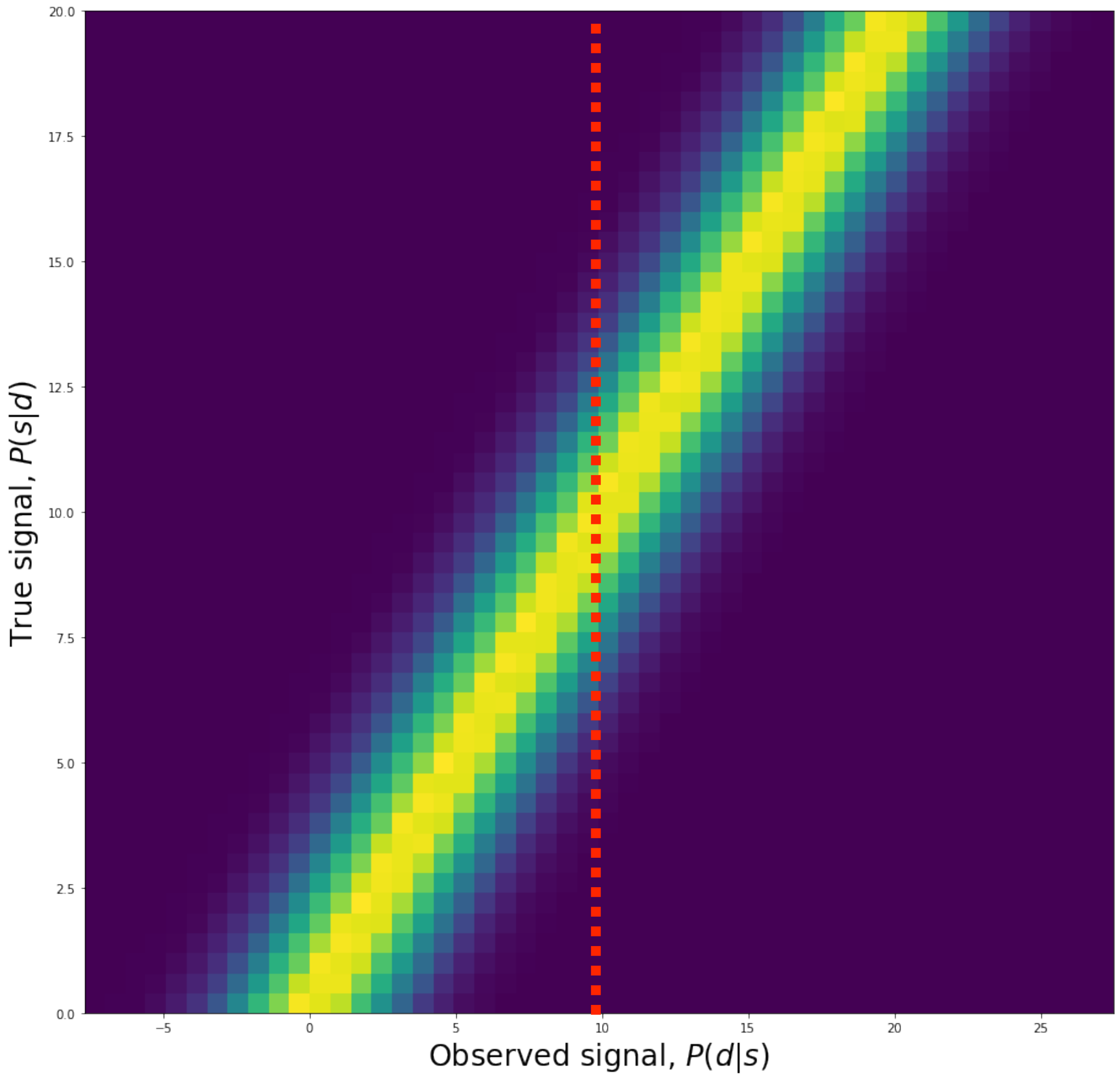
True signal = 10
Observed signal

$$P(d\,|\,s)$$

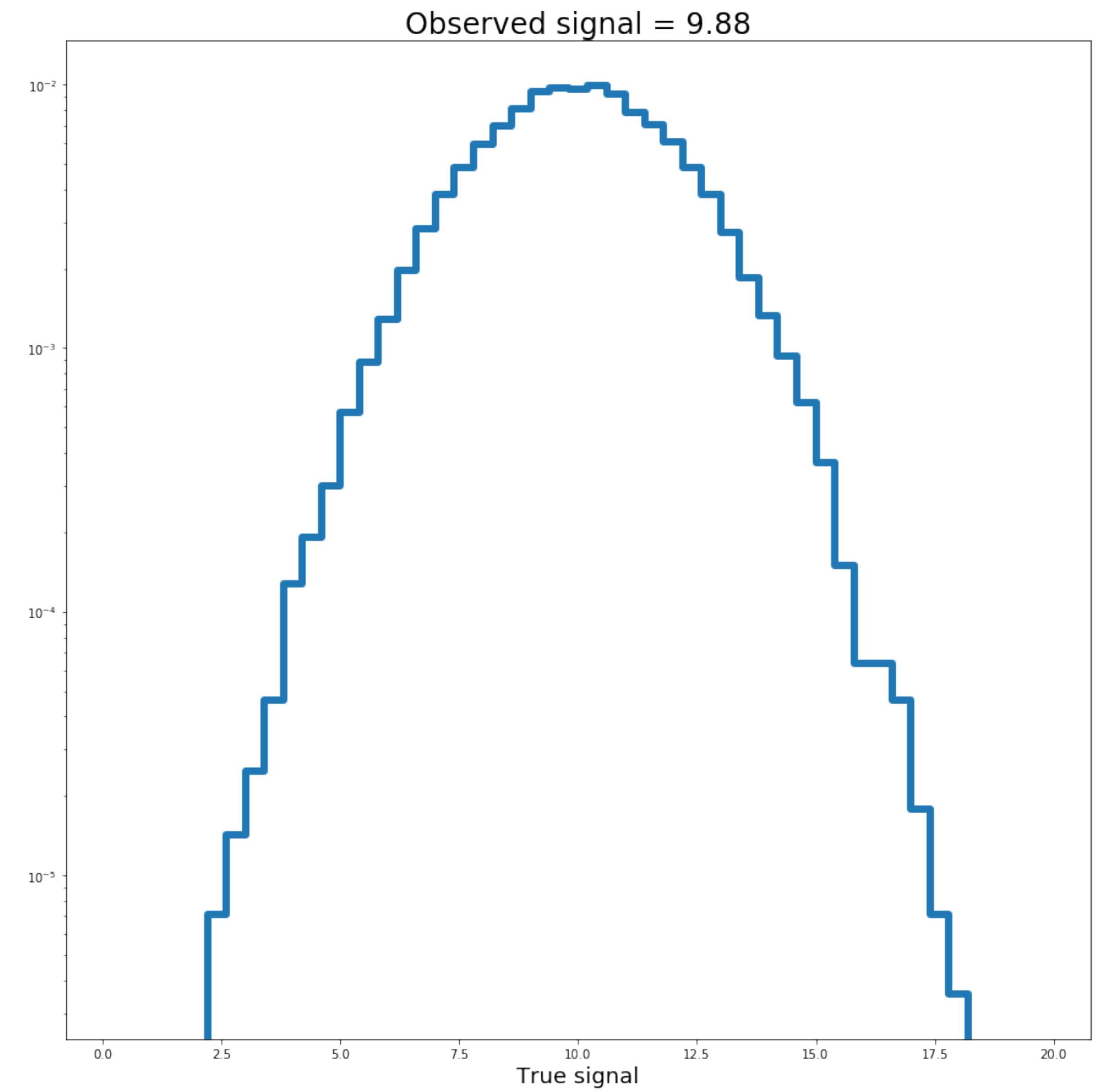
What if I measure data value X and what to know what the range of true signals might be?



Slices



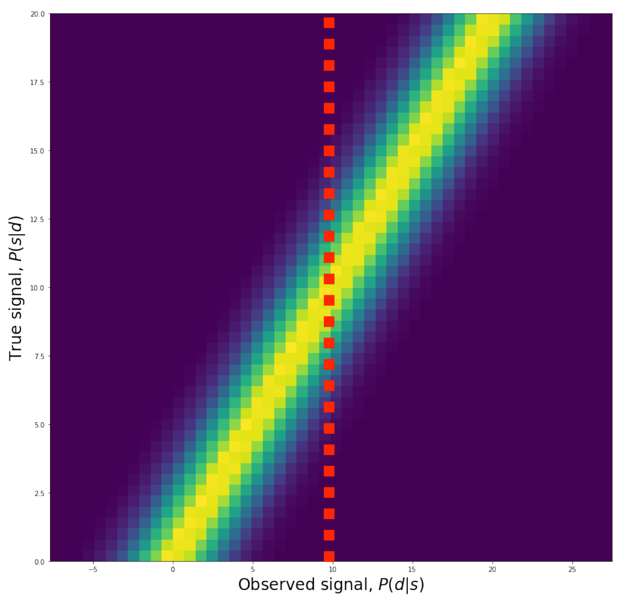
True signal, $P(s|d)$
Observed signal, $P(d|s)$

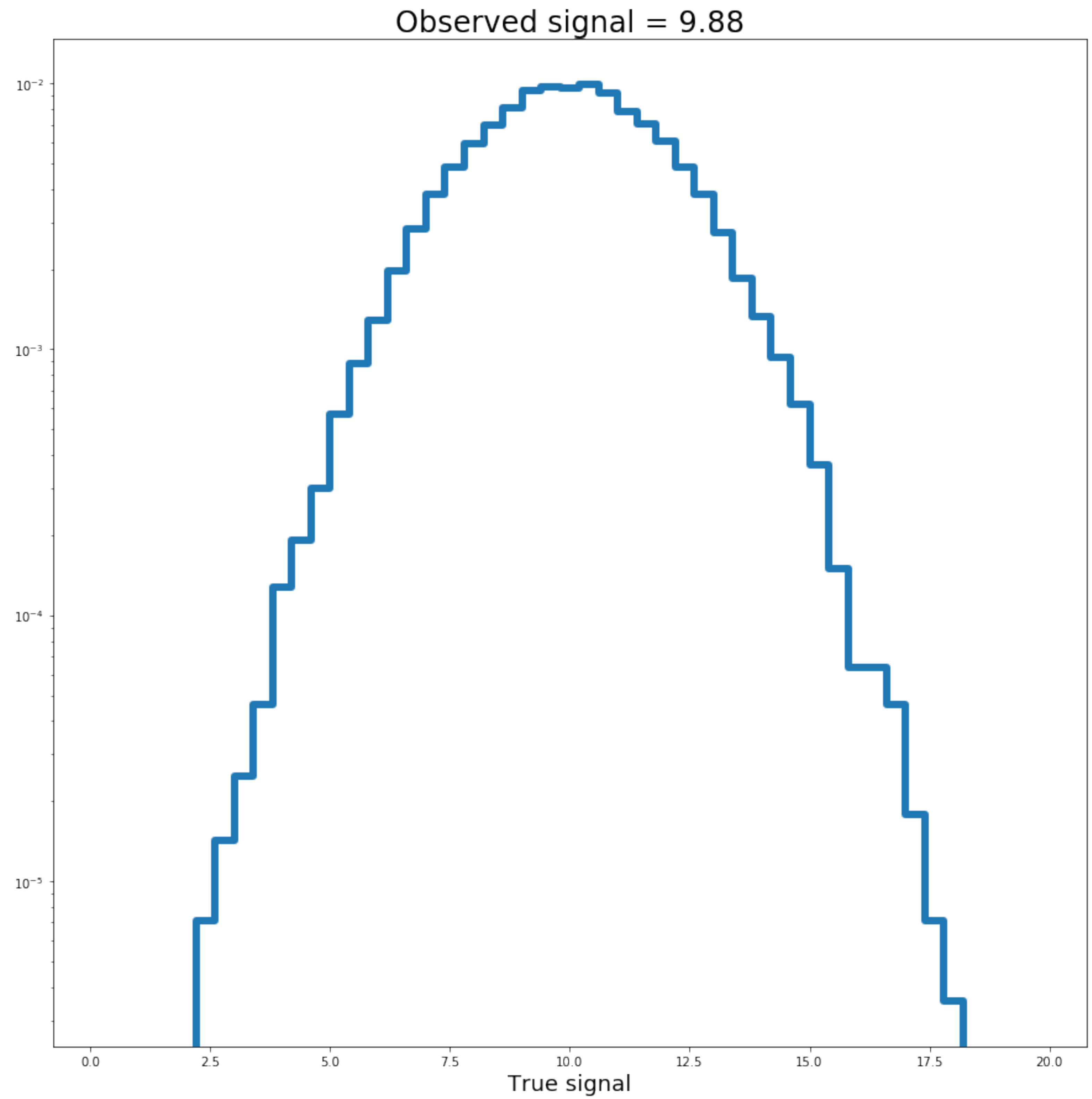


Observed signal = 9.88
True signal

$$P(s\,|\,d)$$

Confidence interval

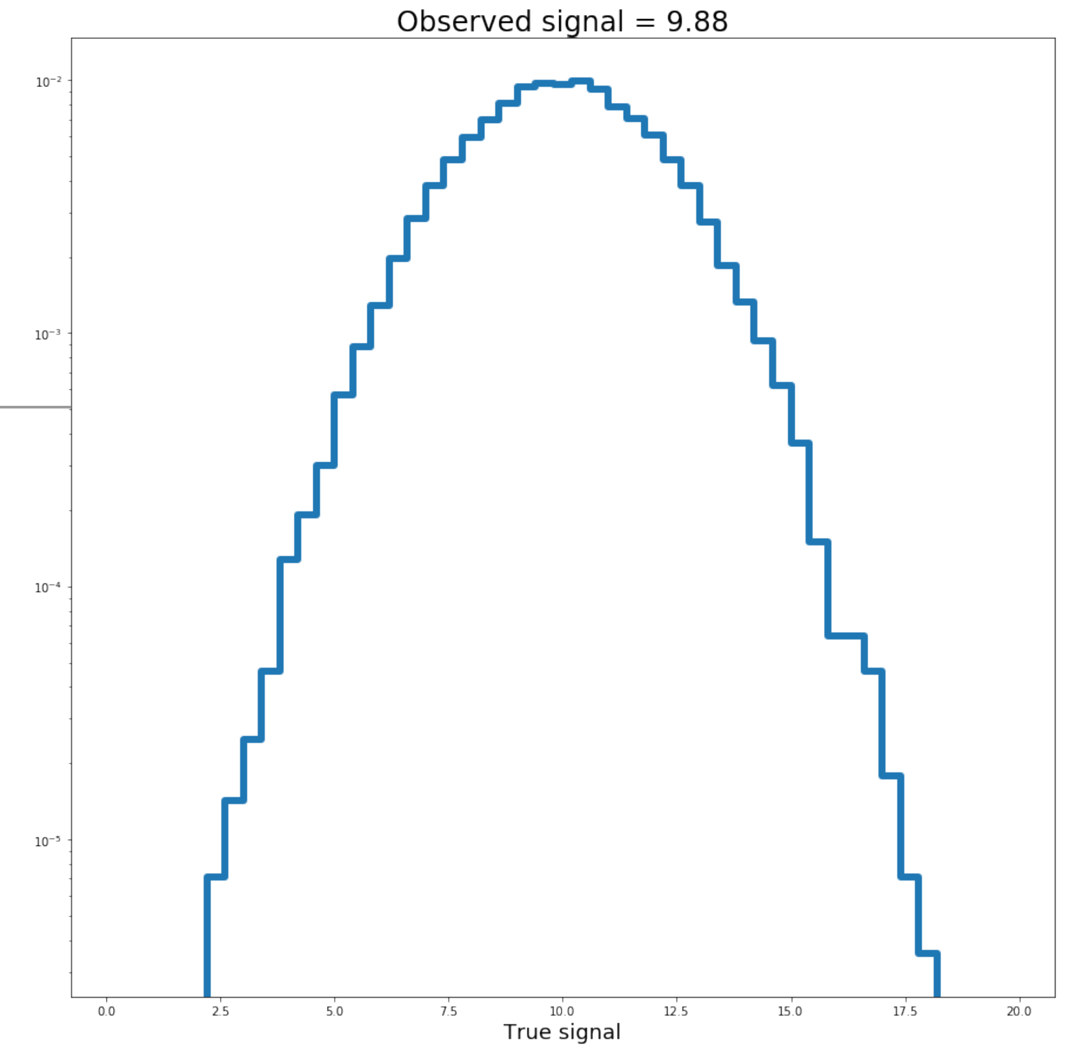




$$9.88 \pm 2.0$$

Confidence interval

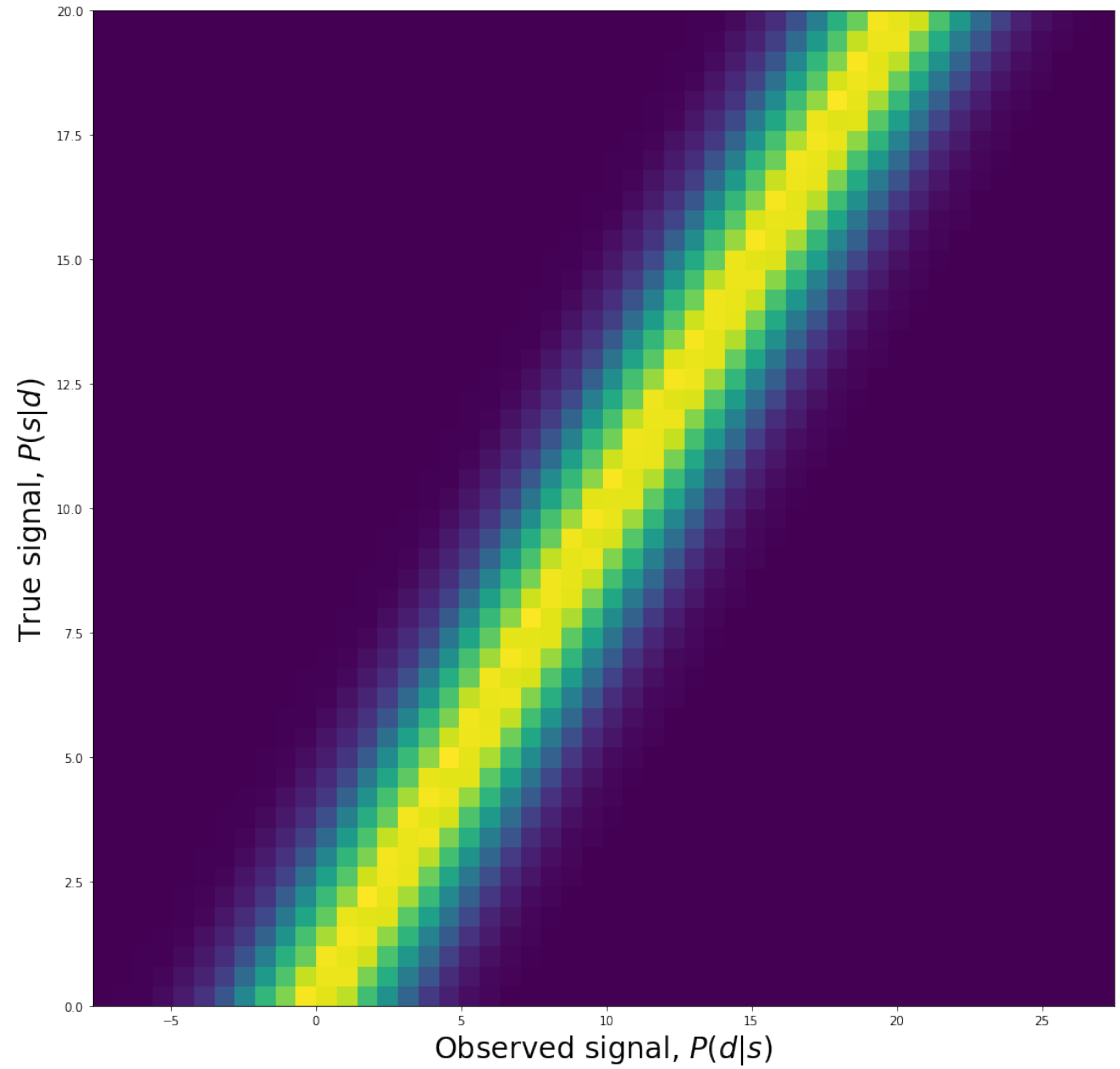
$$\underbrace{9.88} \pm \underbrace{2.0}$$



- Best guess at the true signal
- 1σ: "68% of the time the true signal will be in this range"
- 2σ "95% of the time the true signal will be in this range"

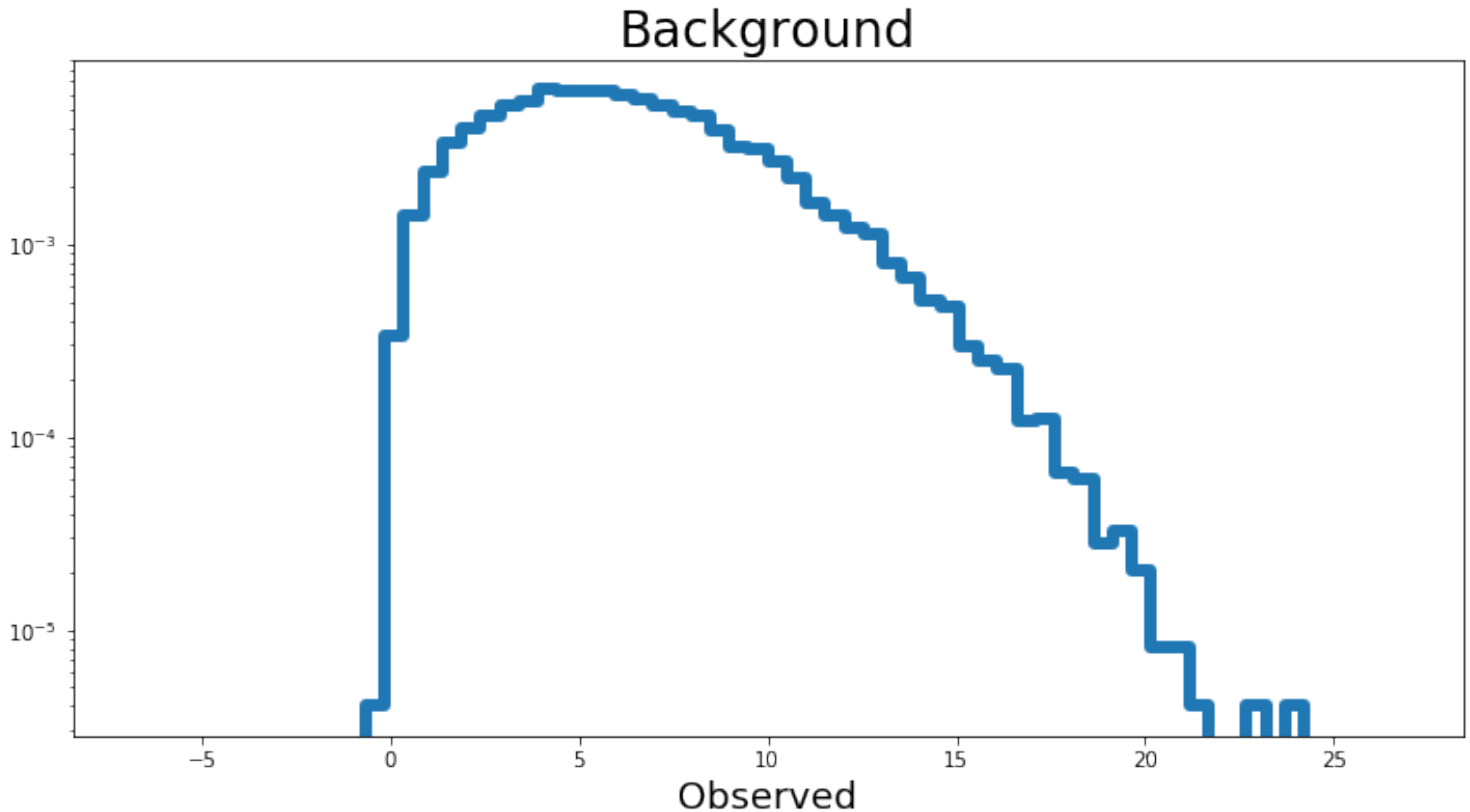
Visual Bayes Theorem

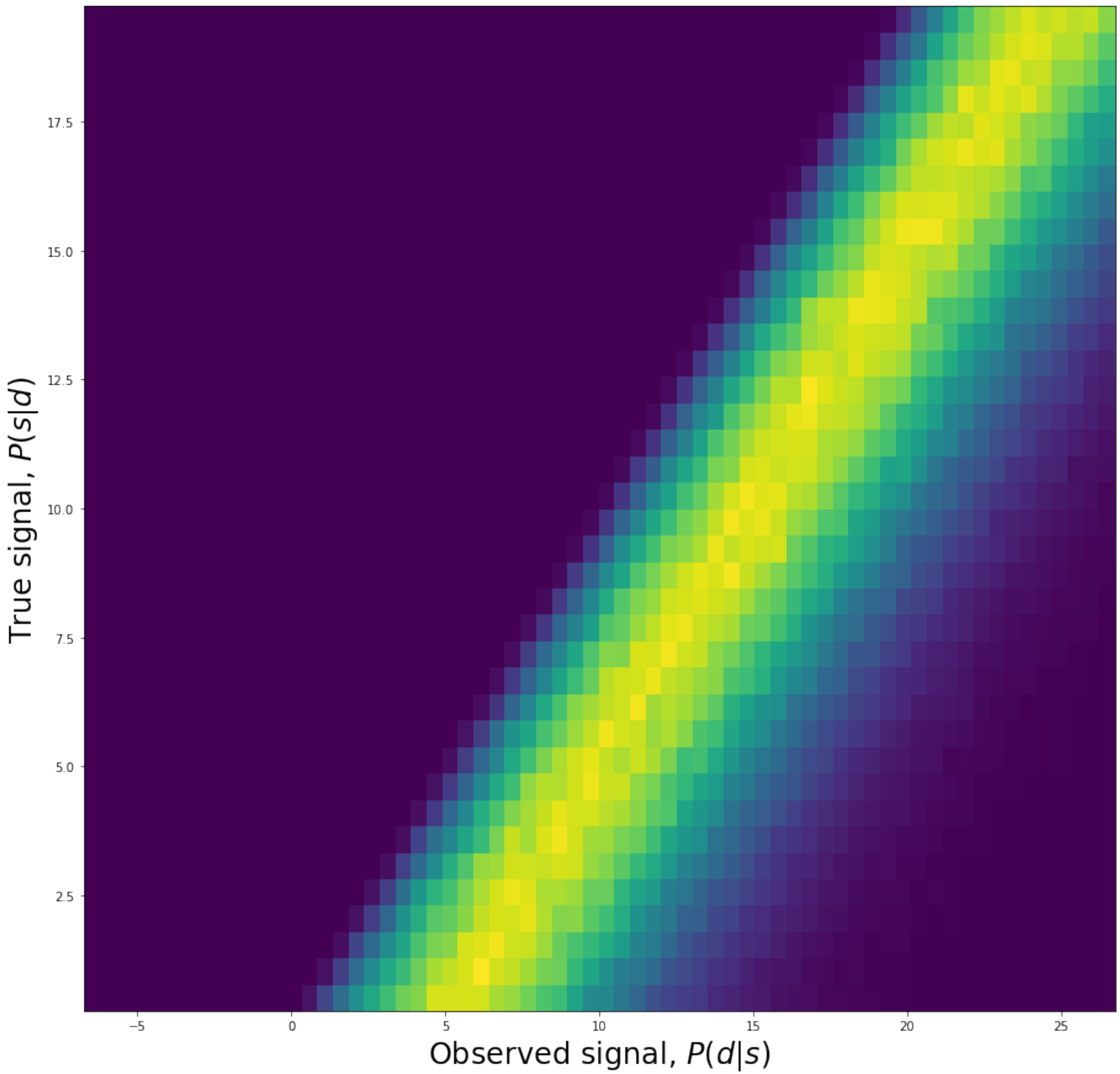
$$P(s\,|\,d) = \frac{P(d\,|\,s)P(s)}{P(d)}$$



Asymmetric, non-zero mean background

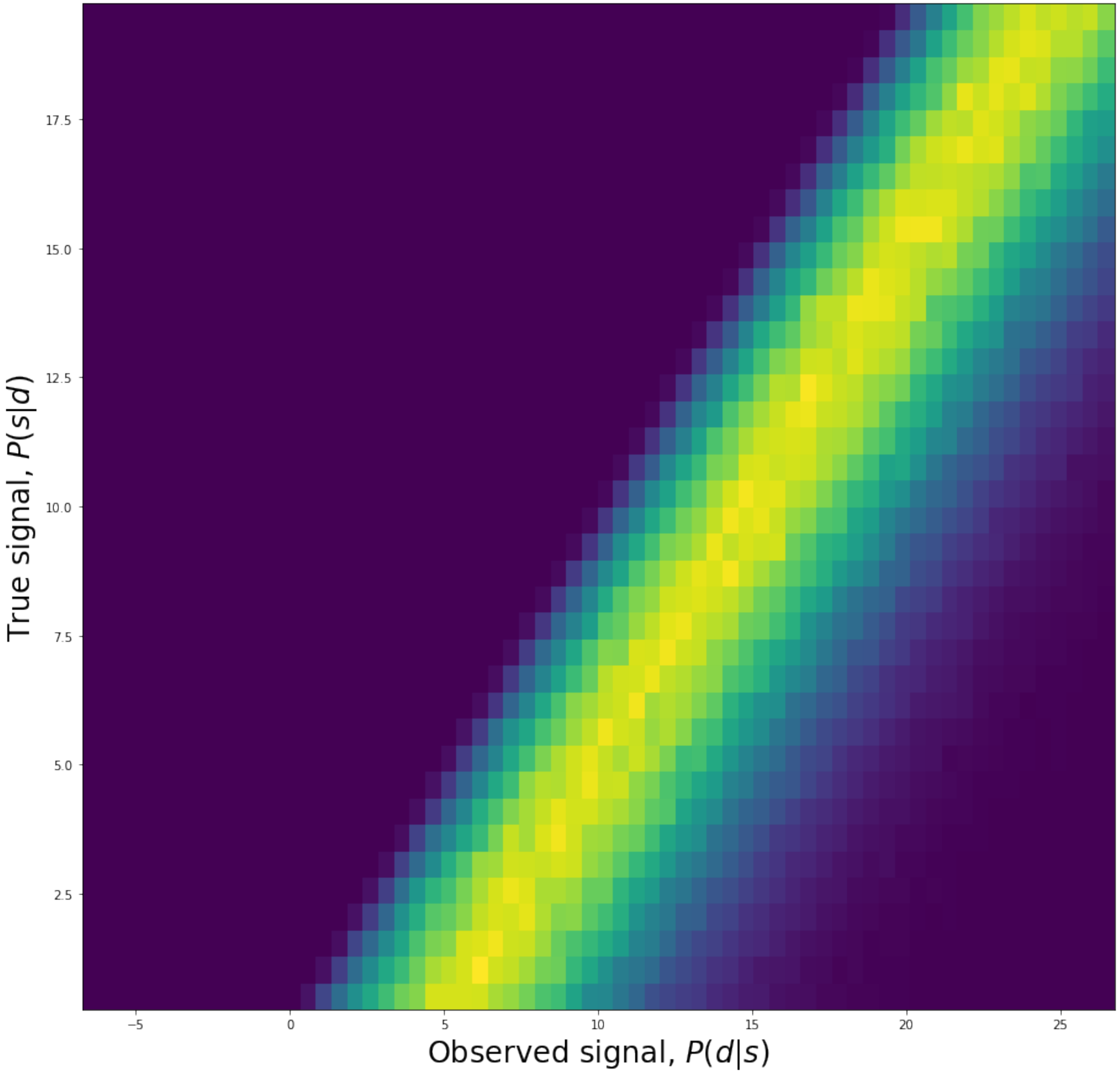
Background

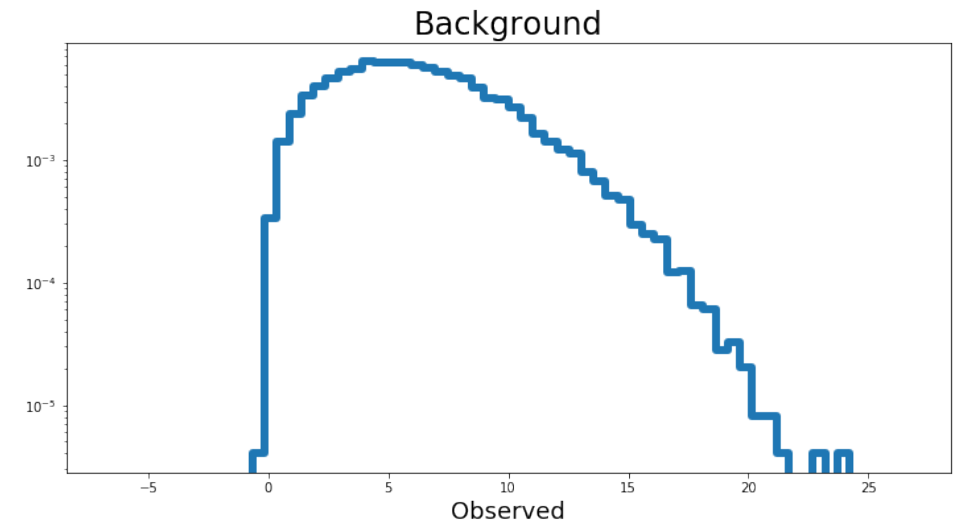




True signal, $P(s|d)$
17.5
15.0
12.5
10.0
7.5
5.0
2.5
−5
0
5
10
15
20
25
Observed signal, $P(d|s)$

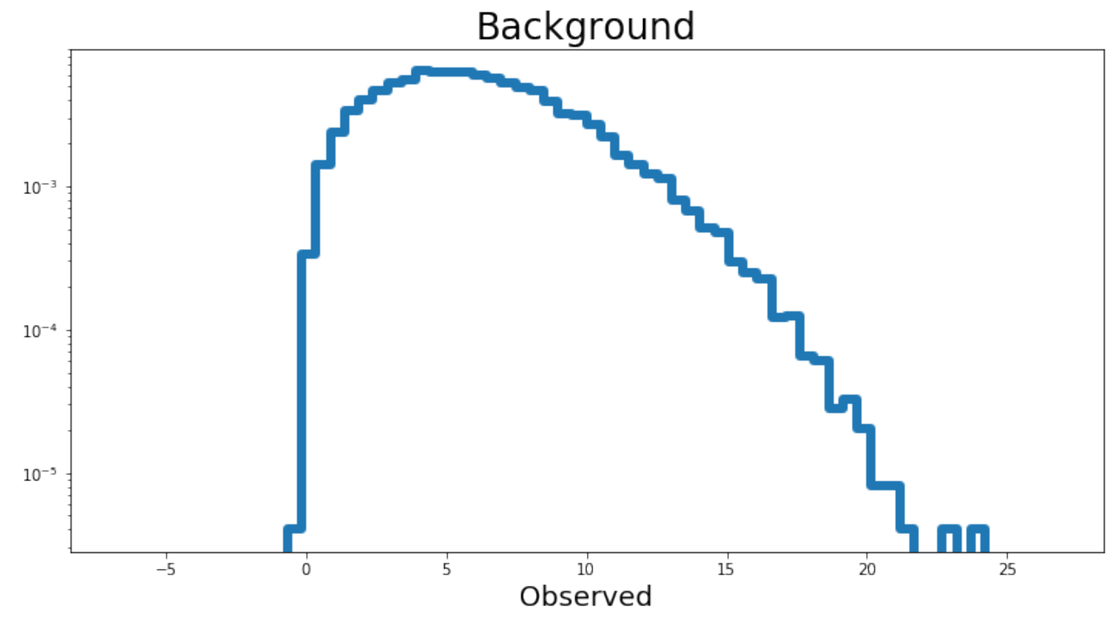
You observe a signal of 14, sketch $P(s \mid d)$

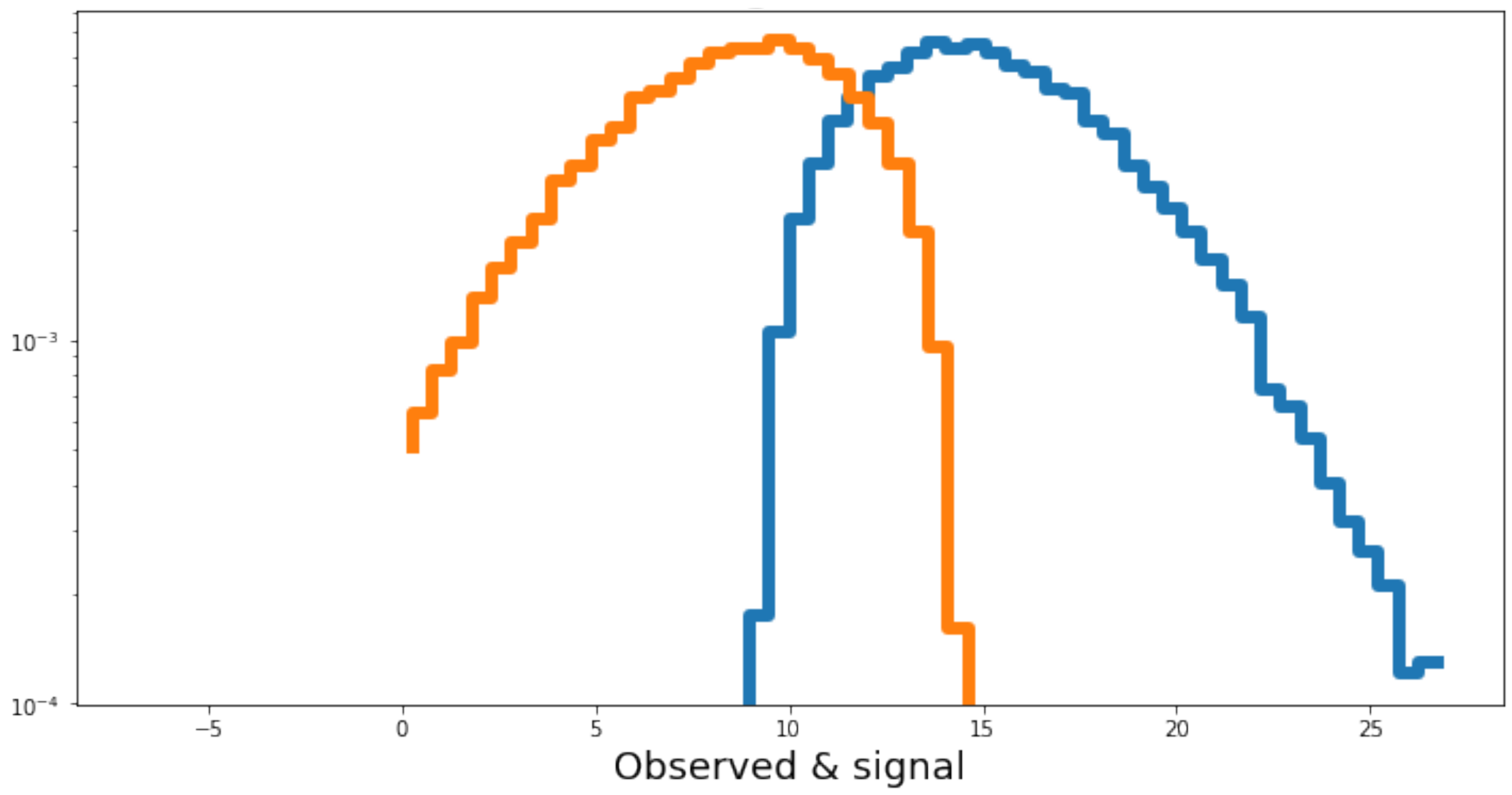


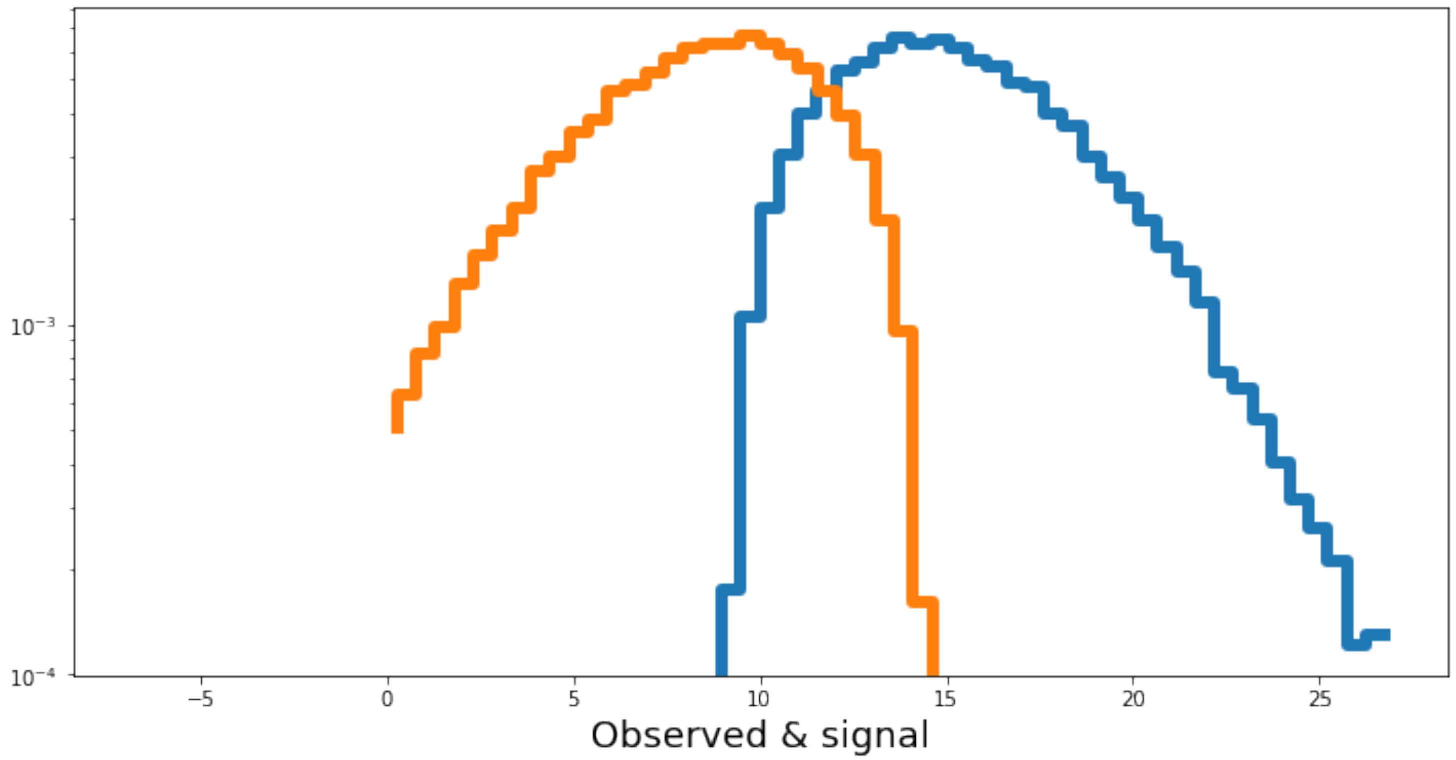


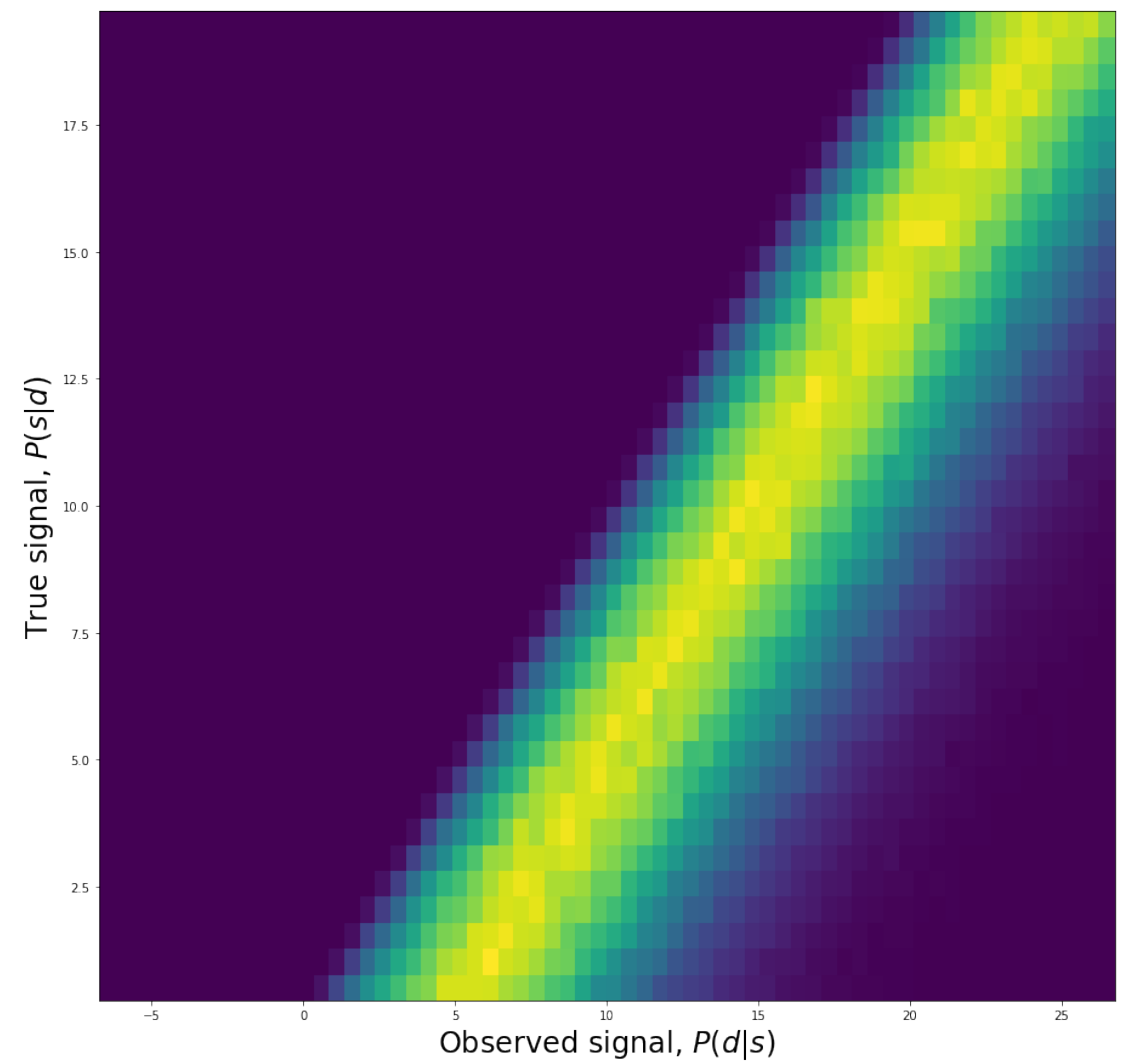
Sketch

- $P(d \mid s = 9.5)$
- $P(s \mid d = 14)$






10^{-3}
10^{-4}
−5
0
5
10
15
20
25
Observed & signal


True signal, $P(s|d)$
17.5
15.0
12.5
10.0
7.5
5.0
2.5
−5
0
5
10
15
20
25
Observed signal, $P(d|s)$

$$9.5^{+2.3}_{-4.1}$$

Reading a reported value

- ± indicates range of true signal 65% of the time (1σ) or 95% of the time (2σ)

- $X \pm Y$ implies symmetric background distribution, usually implies Gaussian

- X^{+z}_{-y} indicates asymmetric background distribution

Upper limits

Upper Limit

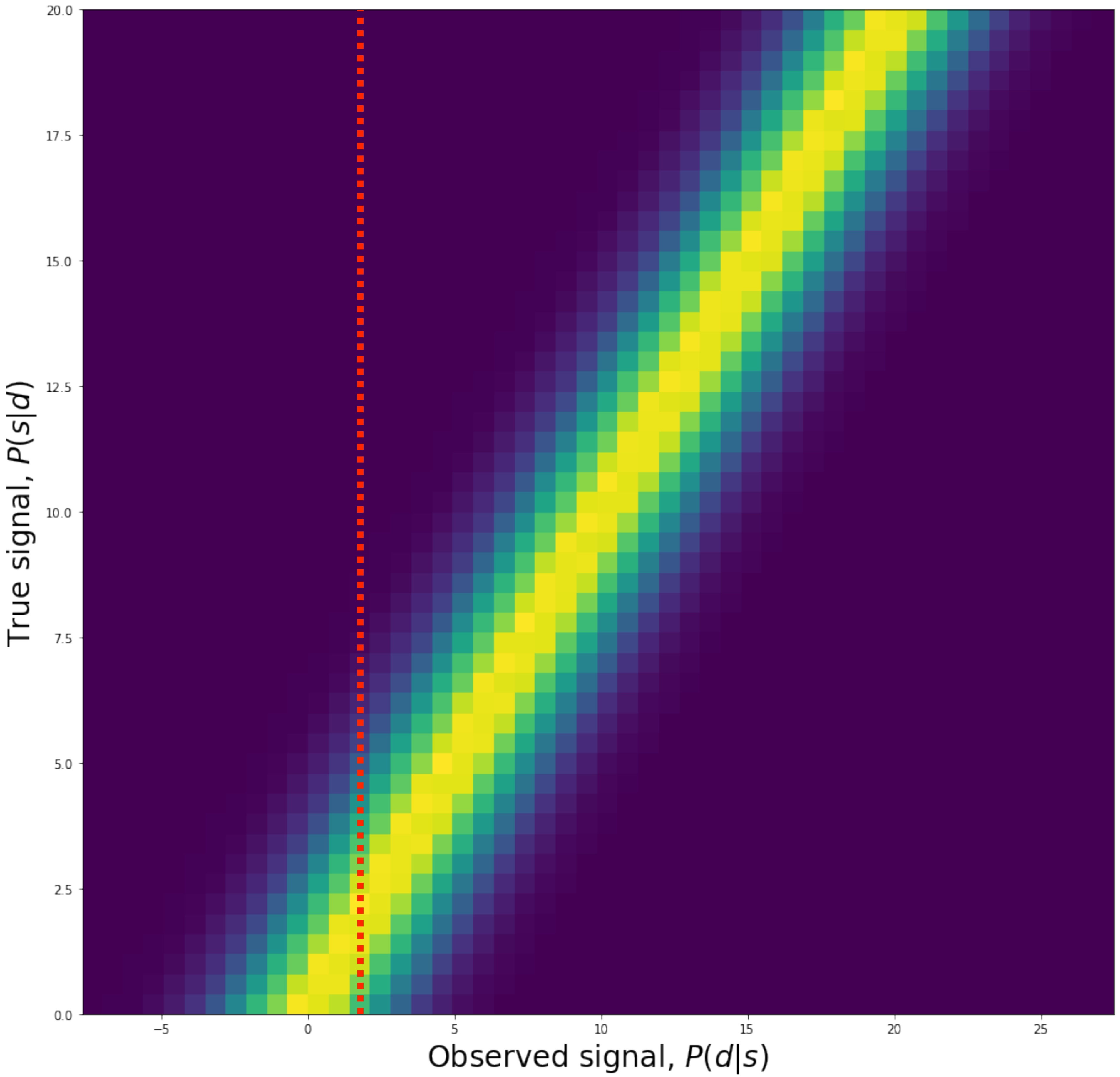
I didn't see anything significant…

- How faint must the signal have been for me not to see it?

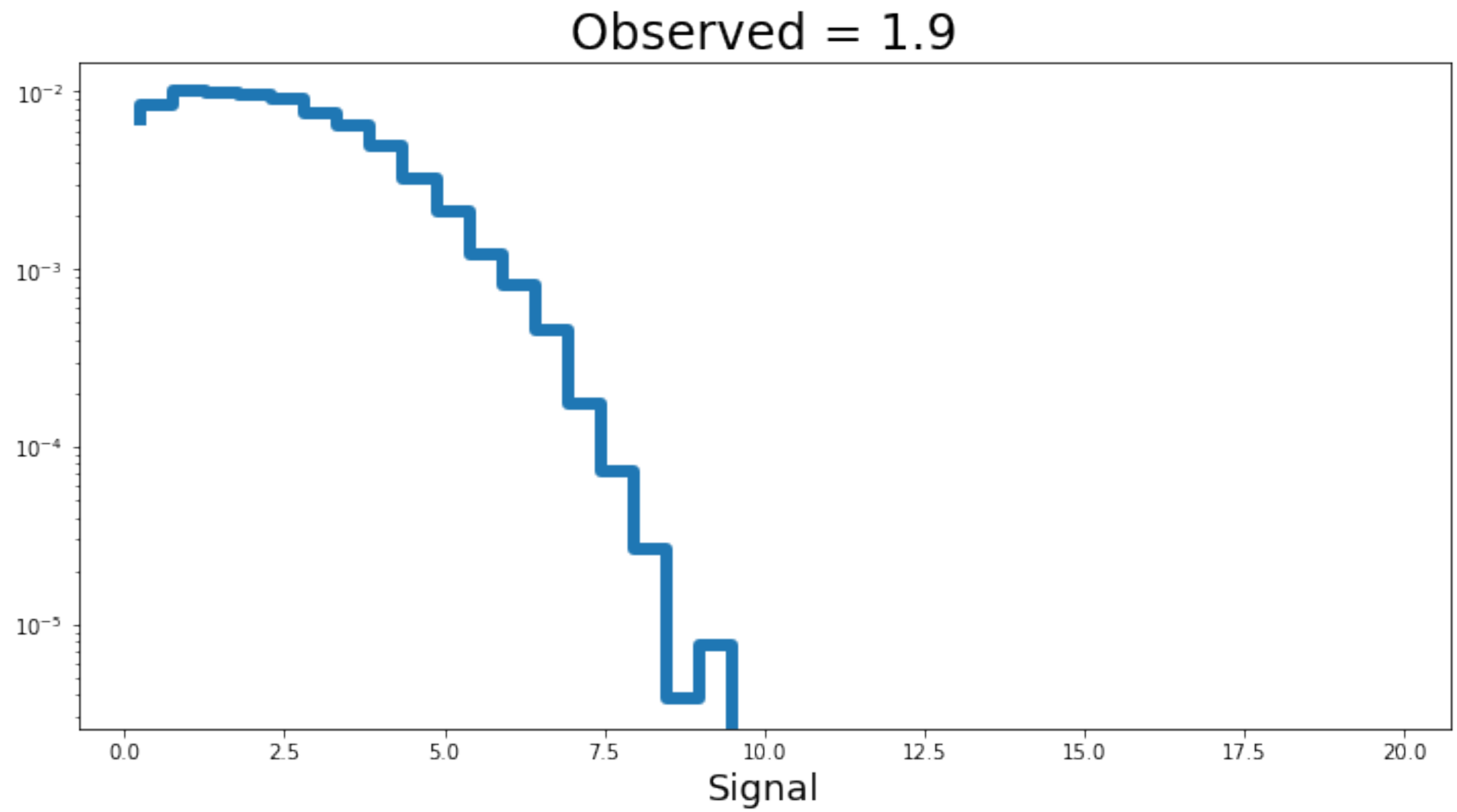
What is the statistical question?

- How faint must the signal have been for me not to see it?
- If it was brighter than X, I would have seen it $95\,\%$ of the time

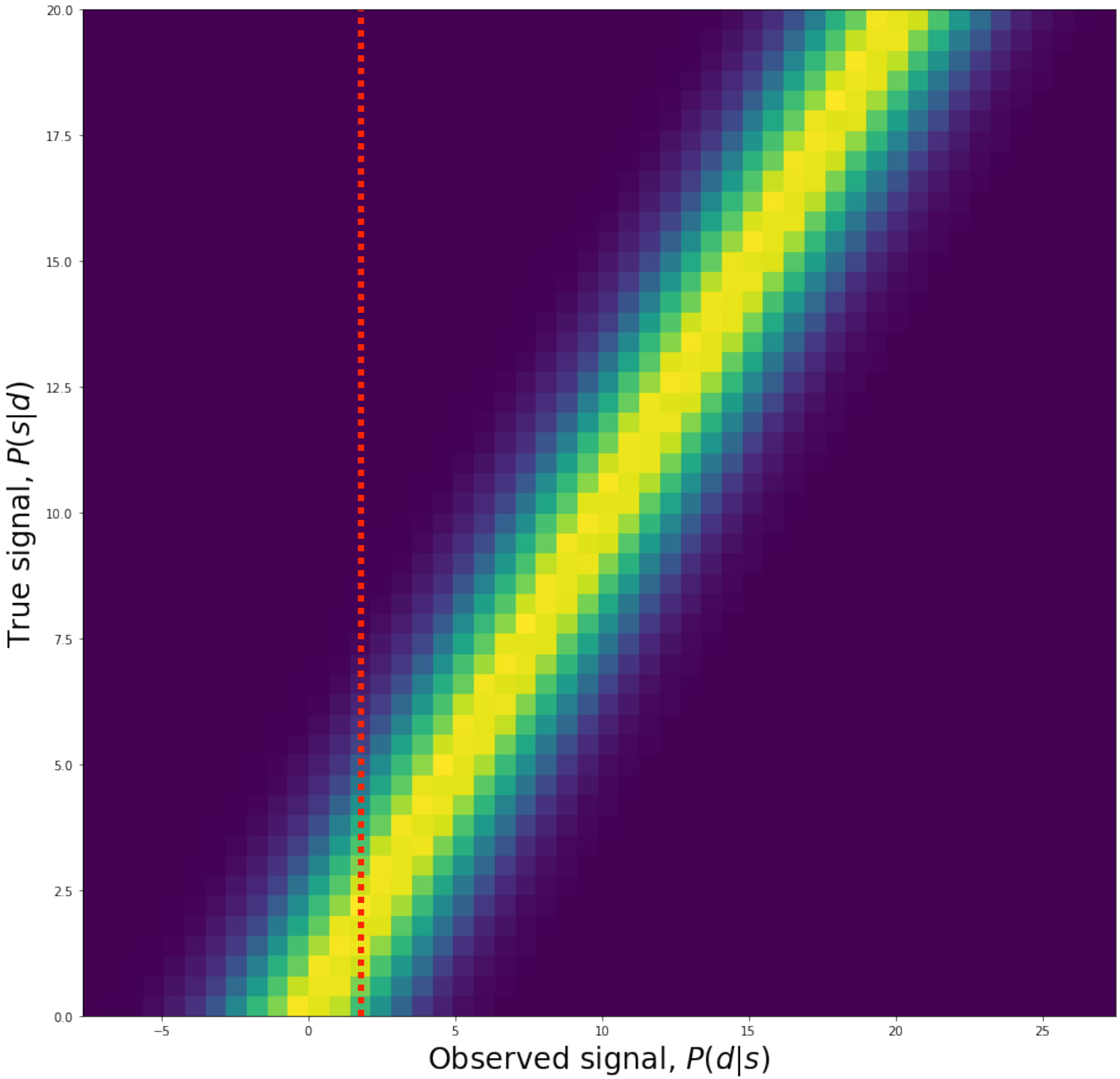
Simulated observations



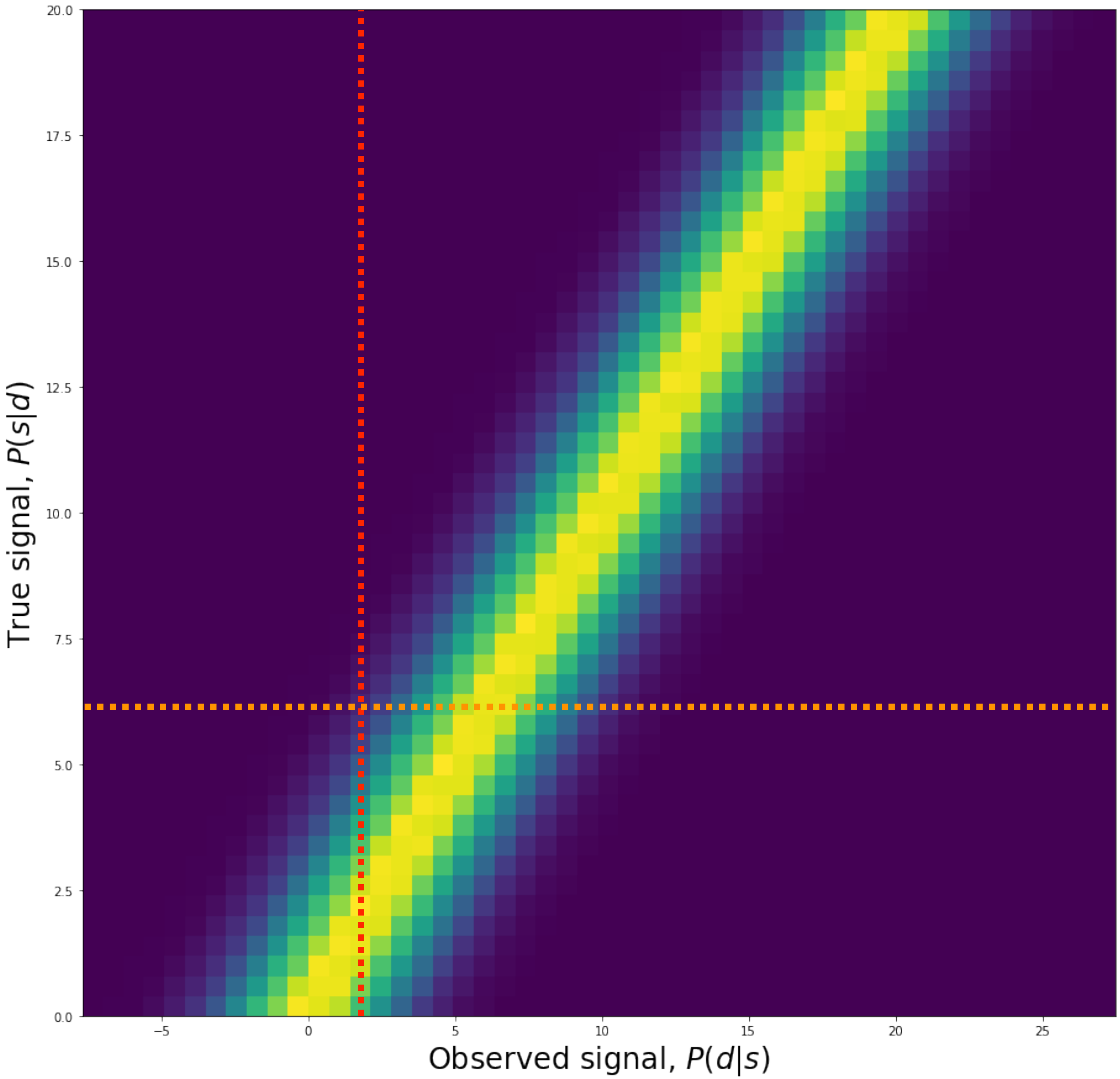
Signal pdf()


Observed = 1.9
10^{-2}
10^{-3}
10^{-4}
10^{-5}
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Signal

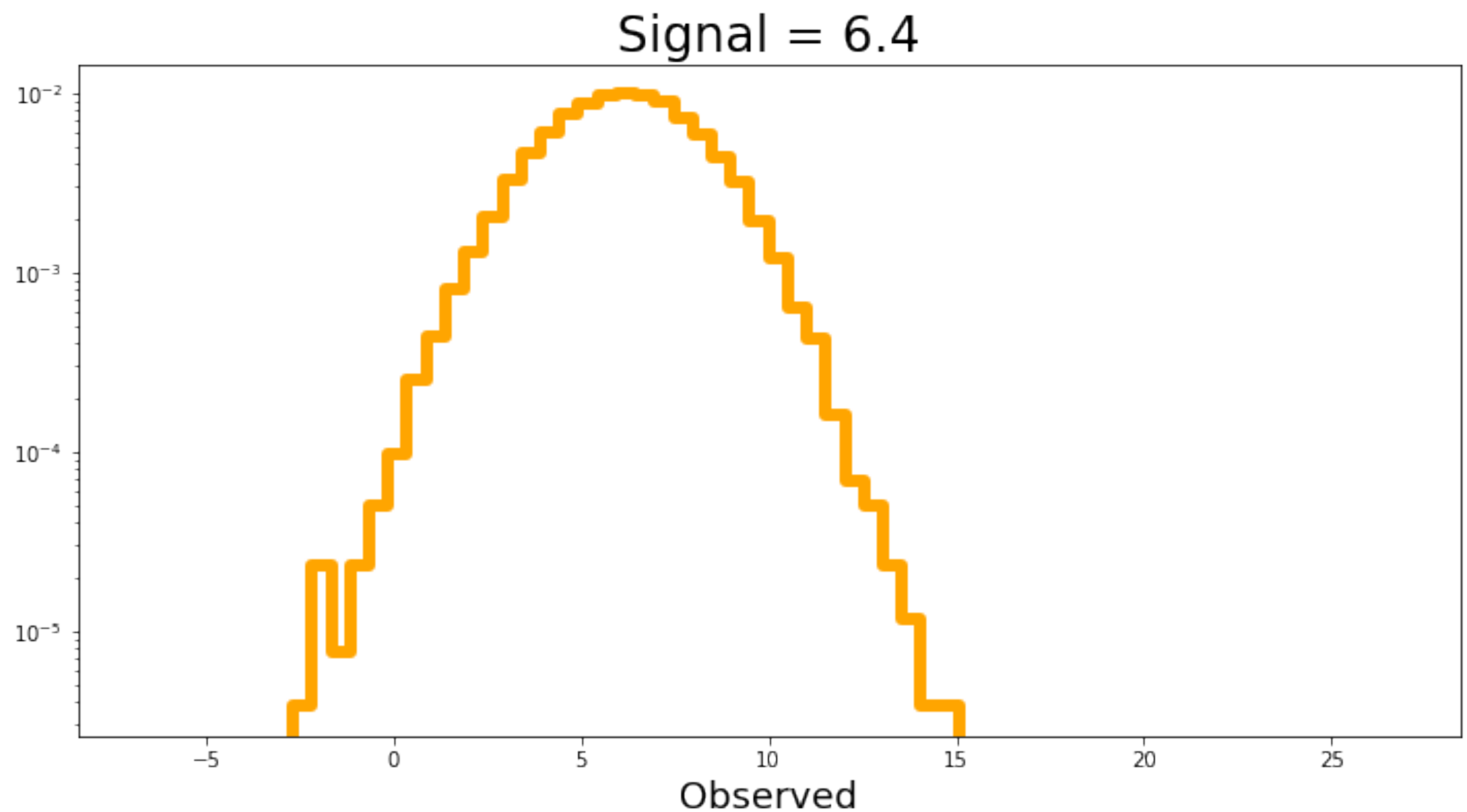
Simulated observations



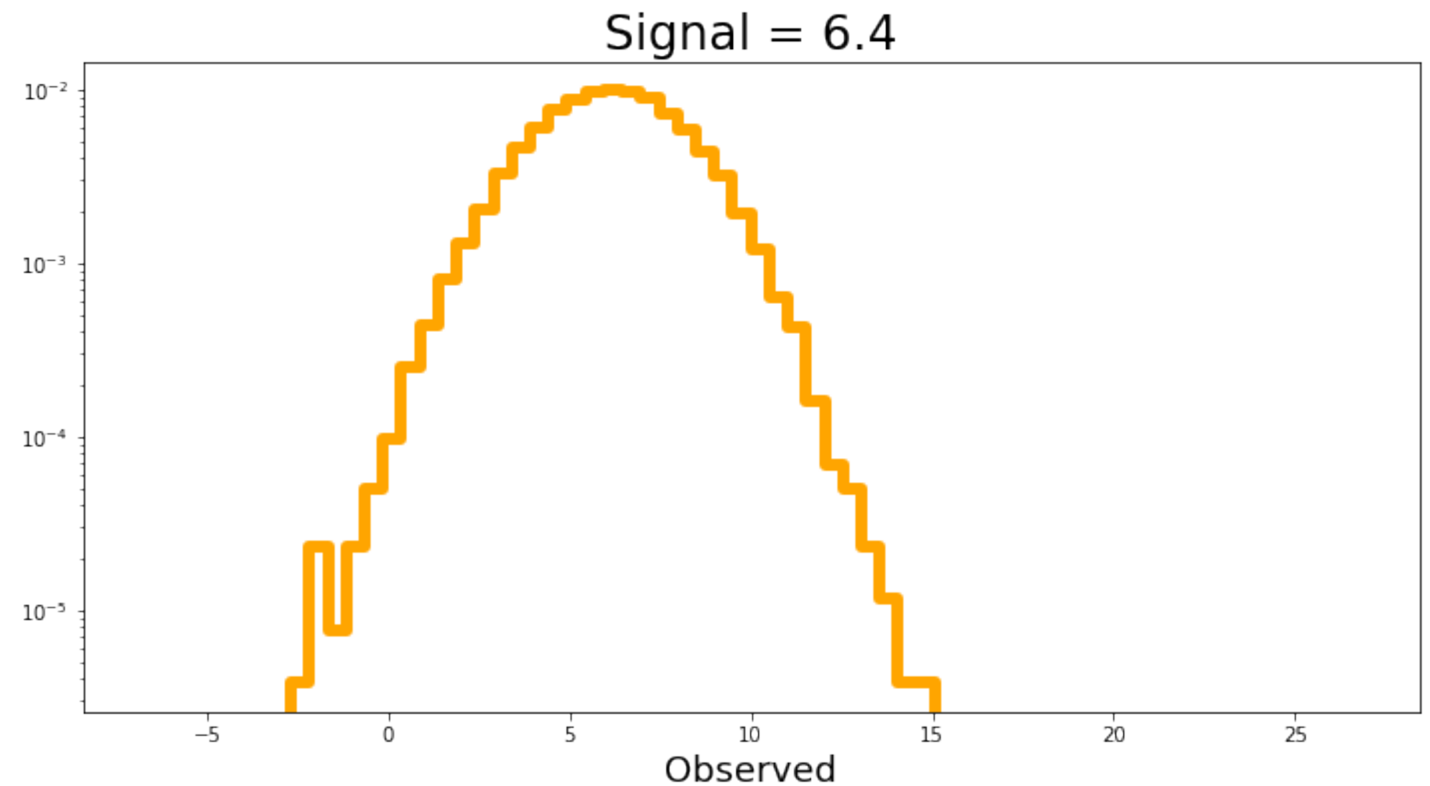
Simulated observations



Signal you would have seen 95% of the time



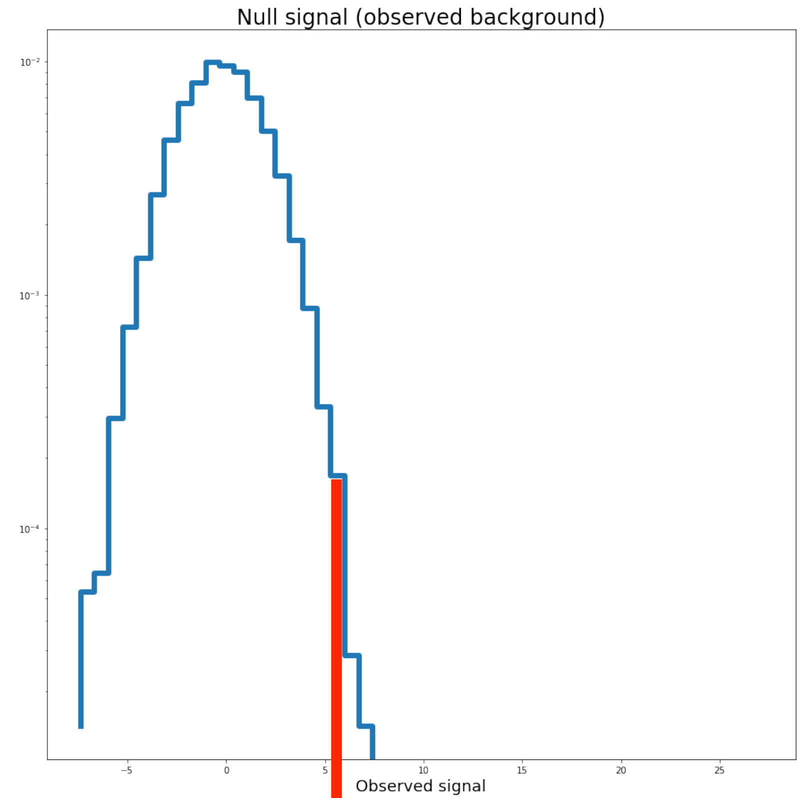
Signal you would have seen 95% of the time



Because I observed a value of 1.9, I know I would have observed a signal of strength 6.4 95% of the time. So my 95% upper limit is 6.4

Putting it all together

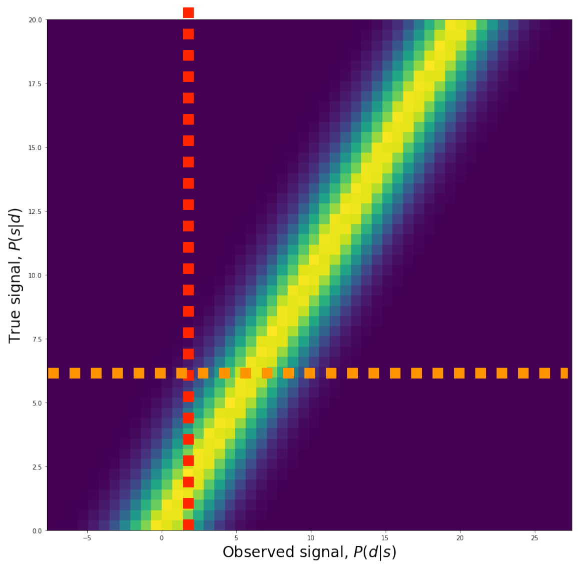
Determine 5σ threshold

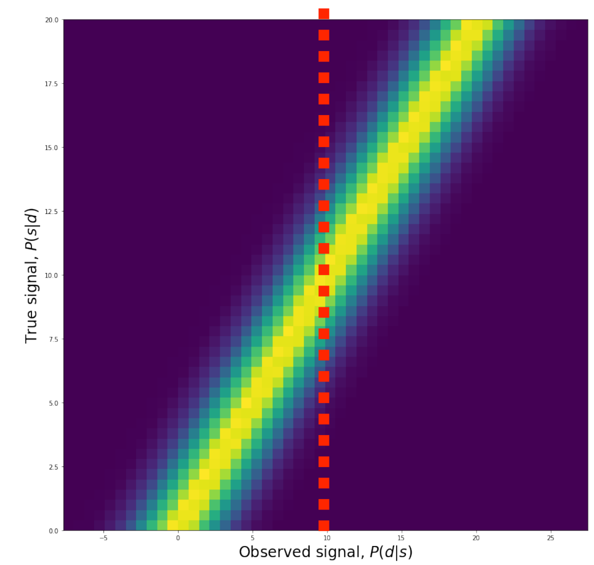


Do experiment

Below 5σ

Above 5σ





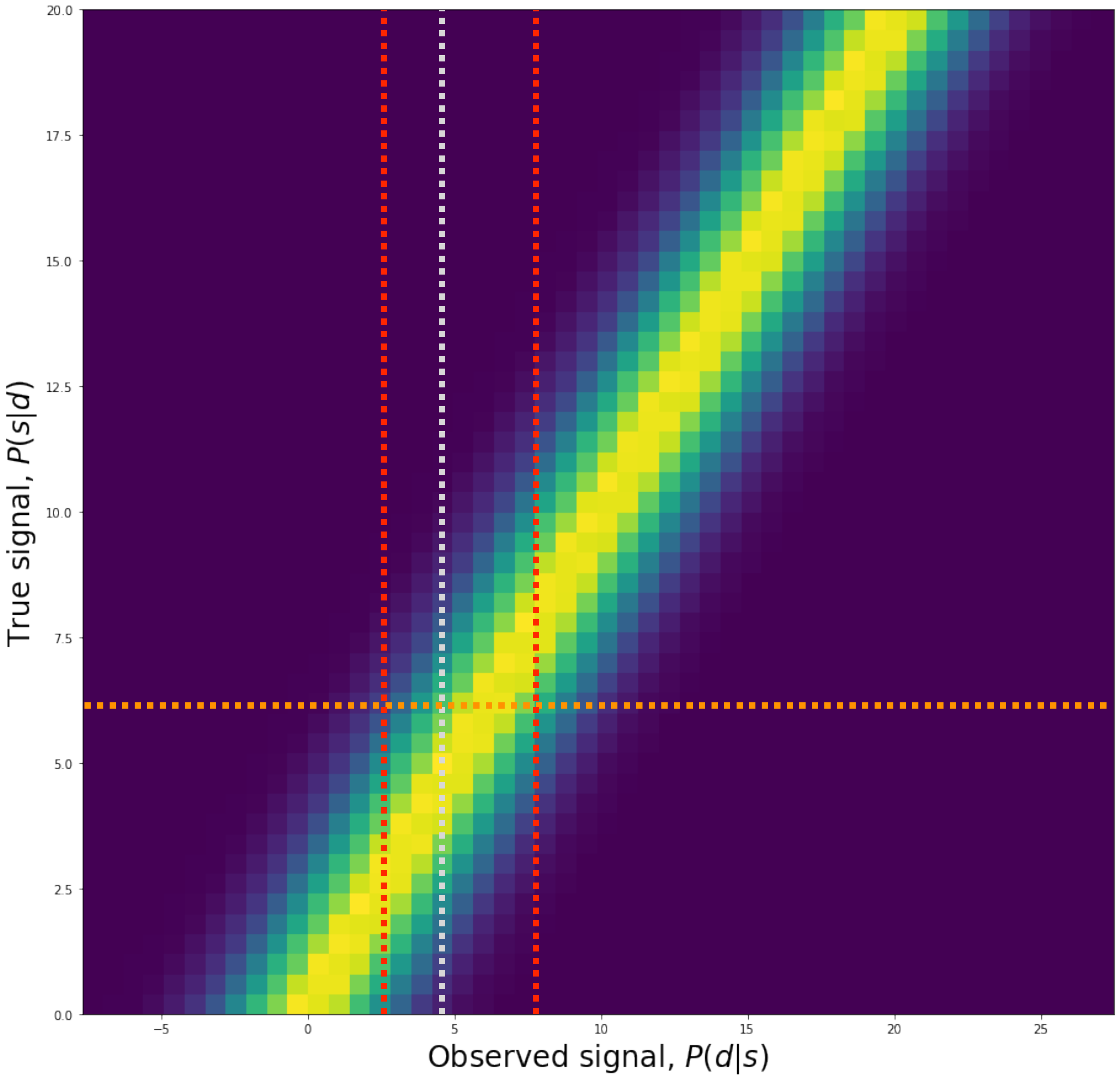
Upper limit

$< X$

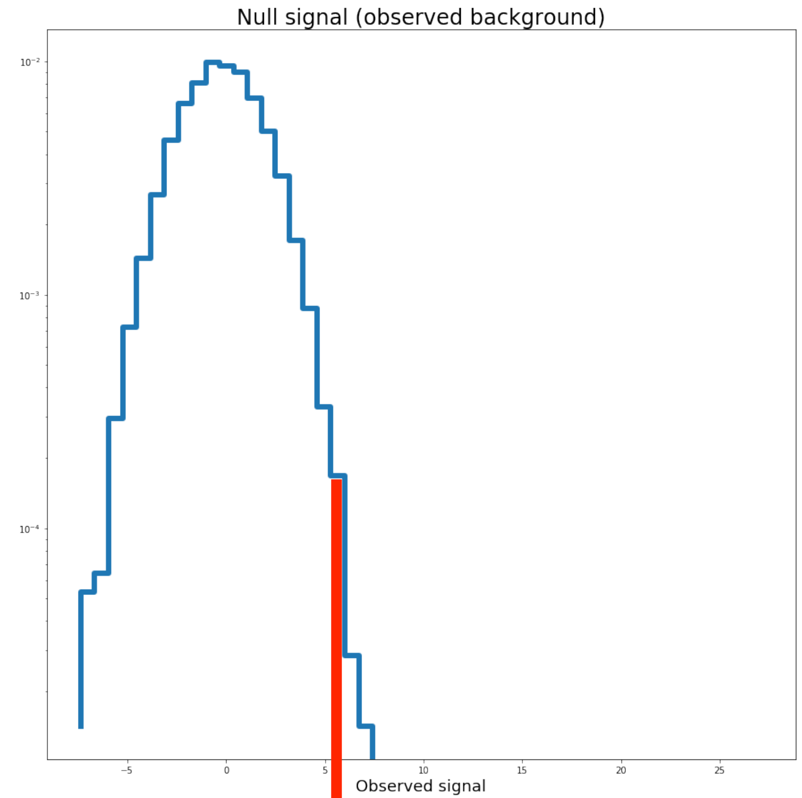
Confidence interval

$X \pm Y$

Simulated observations



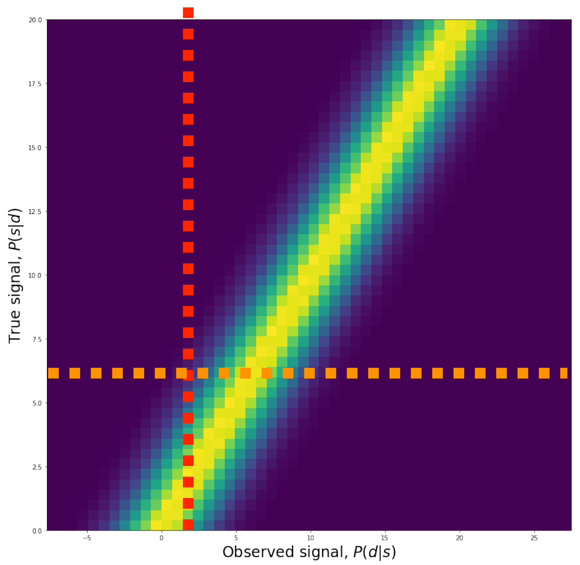
Determine 5σ threshold

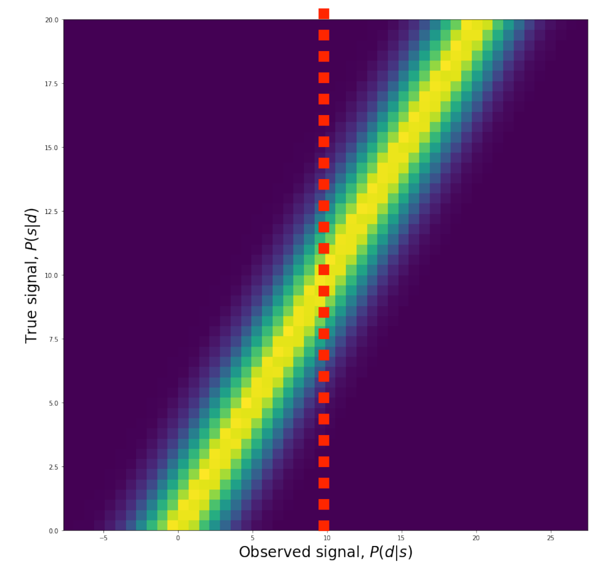


Do experiment

Below 5σ

Above 5σ





Upper limit

$< X$

Confidence interval

$X \pm Y$

No formula

- Determine the question you want to ask in precise words; then convert to math

Not yet questions:

- I’ve done X, what is the significance?
  - What question do you want to ask?
- I have a background ($P(d \mid s_T)$), what prior should I multiply by?
  - ***Very*** dangerous, the Bayesian prior *depends* on the question you want to ask.